Math 1231 Section 10 Fall 2021
Single-Variable Calculus I Mastery Quiz 10
Due Thursday, December 2

This week's mastery quiz has three topics. **You may submit all three**. This is the last week for topic S8.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

You shouldn't spend more than about 20-30 minutes on this quiz. Don't worry if you make a minor error, but try to demonstrate that you understand the concepts involved and have mastered the underlying material. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically through Blackboard but this should be a last resort.

Topics on This Quiz

- Major Topic 5: Integration
- Secondary Topic 8: Riemann Sums
- Secondary Topic 9: Applications of Integrals

Name:

Recitation Section:

M5: Integration

(a) Let $F(x) = \int_1^{3x^2} e^{\sec(t)}\,dt$. What is $F'(x)$?

Solution: If we set $F_1(x) = \int_2^x e^{\sec(t)}\,dt$ then $F_1'(x) = e^{\sec(x)}$, so

$$\frac{d}{dx}F(x) = \frac{d}{dx}F_1(3x^2) = F_1'(3x^2)\cdot 6x = e^{\sec(3x^2)}6x.$$

(b) Compute $\int x\cos(3x^2-2)\,dx =$

Solution: Set $u = 3x^2 - 2$ so $du = 6xdx$ and $dx = \frac{du}{6x}$. Then

$$\begin{aligned}\int x\cos(3x^2-2)\,dx &= \int x\cos(u)\frac{du}{6x} = \frac{1}{6}\int \cos(u)\,du \\ &= \frac{1}{6}\sin(u) + C = \frac{1}{6}\sin(3x^2-2) + C.\end{aligned}$$

(c) $\int_{\pi/12}^{3\pi/8} \csc^2(2t)\,dt =$

Solution: Set $u = 2t$ so $du = 2\,dt$ and $dt = \frac{du}{2}$. Then

$$\begin{aligned}\int_{\pi/12}^{3\pi/8} \csc^2(2t)\,dt &= \int_{\pi/6}^{3\pi/4} \csc^2(u)\frac{du}{2} \\ &= -\frac{1}{2}\cot(u)\Big|_{\pi/6}^{3\pi/4} \\ &= \frac{-\cos(3\pi/4)}{2\sin(3\pi/4)} - \frac{-\cos(\pi/6)}{2\sin(\pi/6)} \\ &= \frac{\sqrt{2}/2}{2\sqrt{2}/2} + \frac{\sqrt{3}/2}{2/2} = \frac{1+\sqrt{3}}{2}.\end{aligned}$$

Secondary Topic 8: Riemann Sums

Let $f(x) = 2x^2$ be defined on the interval $[-2, 1]$.

(a) Approximate the area under the curve of the function using three rectangles and right endpoints.

(b) Approximate the area under the curve of the function using three rectangles and left endpoints.

(c) Write a formula for R_n, the estimate using n rectangles and right endpoints, as a summation of n terms.

(d) Use your answer in part (c) to find a closed-form formula for R_n. (This formula should not have a summation sign or be given as a sum of n terms.)

(e) Use the formula in part (c) to compute the area exactly.

Solution:

(a) $R_3 = 1 \cdot f(-1) + 1 \cdot f(0) + 1 \cdot f(1) = 2 + 0 + 2 = 4.$

(b) $L_3 = 1 \cdot f(-2) + 1 \cdot f(-1) + 1 \cdot f(0) = 8 + 2 + 0 = 10.$

(c)

$$R_n = \sum_{i=1}^{n} \frac{3}{n} f\left(-2 + \frac{3i}{n}\right) = \frac{3}{n} \sum_{i=1}^{n} 2(-2 + 3i/n)^2.$$

(d)

$$\begin{aligned} R_n &= \frac{3}{n} \sum_{i=1}^{n} 2(-2+3i/n)^2 \\ &= \frac{6}{n} \sum_{i=1}^{n} (4 - 12i/n + 9i^2/n^2) \\ &= \frac{6}{n} \left(\sum_{i=1}^{n} 4 + \sum_{i=1}^{n} -12i/n + \sum_{i=1}^{n} 9i^2/n^2 \right) \\ &= \frac{24}{n} \sum_{i=1}^{n} 1 - \frac{72}{n^2} \sum_{i=1}^{n} i + \frac{54}{n^3} \sum_{i=1}^{n} i^2 \\ &= \frac{24}{n} \cdot n - \frac{72}{n^2} \frac{n(n+1)}{2} + \frac{54}{n^3} \frac{n(n+1)(2n+1)}{6} \end{aligned}$$

(e) We can compute

$$\begin{aligned} \lim_{n \to +\infty} R_n &= \lim_{n \to +\infty} \frac{24}{n} \cdot n - \frac{72}{n^2} \frac{n(n+1)}{2} + \frac{54}{n^3} \frac{n(n+1)(2n+1)}{6} \\ &= 24 - 36 + 18 = 6. \end{aligned}$$

S9: Applications of Integrals

(a) Suppose your velocity is given by the function $v(t) = 3\cos(\pi t/2) + 1$, and your position at time 0 is 3. What is your position at time 5?

Solution:

We can write $v(t) = p'(t)$. Then we have $p(0) = 3$ and we want to find $p(5)$; we know that

$$\begin{aligned} p(5) - p(0) &= \int_0^5 p'(t)\,dt = \int_0^5 v(t)\,dt \\ &= \int_0^5 3\cos(\pi t/2) + 1\,dt = \frac{6}{\pi}\sin(\pi t/2) + t\Big|_0^5 \\ &= \frac{6}{\pi}\sin(5\pi/2) + 5 - \frac{6}{\pi}\sin(0) - 0 = \frac{6}{\pi} + 5. \end{aligned}$$

Thus your final position is $p(5) = \frac{6}{\pi} + 8$.

We could also compute the integral via an explicit u substitution. We'd take $u = \pi t/2$ so that $du = \pi\,dt/2$ and $dt = 2\,du/\pi$, and so

$$\begin{aligned} p(5) - p(0) &= \int_0^5 3\cos(\pi t/2) + 1\,dt = \int_0^{5\pi/2} (3\cos(u) + 1)\,\frac{2\,du}{\pi} \\ &= \int_0^{5\pi/2} \frac{6}{\pi}\cos(u) + \frac{2}{\pi}\,du \\ &= \frac{6}{\pi}\sin(u) + \frac{2u}{\pi}\Big|_0^{5\pi/2} \\ &= \frac{6}{\pi}\sin(5\pi/2) + \frac{2\cdot\frac{5\pi}{2}}{\pi} - \frac{6}{\pi}\sin(0) - \frac{2\cdot 0}{\pi} \\ &= \frac{6}{\pi} + 5 - 0 - 0 = \frac{6}{\pi} + 5. \end{aligned}$$

Again, we conclude that $p(5) = \frac{6}{\pi} + 8$.

(b) What is the average value of the function $f(x) = x^3 - x$ on the interval $[0, 2]$?

Solution:

$$\frac{1}{2}\int_0^2 x^3 - x\,dx = \frac{1}{2}\frac{x^4}{4} - \frac{x^2}{2}\Big|_0^2 = \frac{1}{2}\left(4 - 2 - (0 - 0)\right) = 1.$$