Math 1231 Section 16 Fall 2021
Single-Variable Calculus I Mastery Quiz 10
Due Monday, November 22

This week's mastery quiz has three topics. You may submit all three.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

You shouldn't spend more than about 20-30 minutes on this quiz. Don't worry if you make a minor error, but try to demonstrate that you understand the concepts involved and have mastered the underlying material. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically through Blackboard but this should be a last resort.

Topics on This Quiz

- Major Topic 4: Optimization
- Secondary Topic 7: Approximation
- Secondary Topic 8: Riemann Sums

Name:

Recitation Section:

Major Topic 4: Optimization

(a) A poster needs to have an area of $216in^2$, with 1-inch margins on the bottom and sides and a 2-inch margin on the top. What dimension maximize the area of the *printed* region, excluding margins? **Solution:** Our constraint is that $h \cdot w = 216$, and we want to maximize $(h-3)(w-2)$. So we have $h = 216/w$, and thus we want to maximize

$$\begin{aligned} A(w) &= \left(\frac{216}{w} - 3\right)(w-2) = 180 - 3w - \frac{432}{w} + 6 \\ A'(w) &= -3 + \frac{432}{w^2} \\ w^2 &= 144 \\ w &= \pm 12. \end{aligned}$$

Obviously the only relevant critical point is the positive one, 12.

We can check that this is truly a maximum by the second derivative: $A''(w) = \frac{-720}{w^3} < 0$ for any $w > 0$, so we have a local maximum.

Or we can see that $A'(x) > 0$ when w is small and $A'(w) < 0$ when w is large, so the function is increasing until 80 and decreasing after.

So the dimensions we want are a poster with width 12 inches and height $216/12 = 18$ inches.

(b) Find all the critical points of $g(x) = \dfrac{x^2-8}{x+3}$.

Solution: The function is undefined at $x = -3$.

$g'(x) = \frac{2x(x+3)-1(x^2-8)}{(x+3)^2} = \frac{x^2+6x+8}{(x+3)^2}$. The denominator is zero when $x = -3$, and thus the derivative is undefined there, but so is the function. The numerator is $(x+2)(x+4)$ and thus has roots when $x = -2, -4$. So the critical points of the function are $-2, -3, -4$.

Secondary Topic 7: Approximation

(a) If $f(x) = \sqrt{x^2+1}$, use a quadratic approximation centered at 0 to estimate $f(.2)$.

Solution:

We have $f'(x) = \frac{x}{\sqrt{x^2+1}}$ and $g''(x) = \frac{\sqrt{x^2+1} - \frac{x^2}{\sqrt{x^2+1}}}{x^2+1}$. So $f'(0) = 0$ and $f''(0) = 1$, and then we have

$$\begin{aligned} f(x) &\approx f(0) + f'(0)(x-0) + \frac{f''(0)}{2}(x-0)^2 = 1 + 0x + \frac{1}{2}x^2 = 1 - x^2/2 \\ f(.1) &\approx 1 + .2^2/2 = 1.02. \end{aligned}$$

(b) Use two iterations of Newton's Method to estimate a solution to $x^3 + 3x + 3 = 0$, starting with $x_0 = 0$.

Solution:

We start with $x_0 = 0$. We have $g'(x) = 3x^2 + 3$, and so

$$x_1 = 0 - \frac{3}{3} = -1$$
$$x_2 = -1 - \frac{-1}{6} = \frac{-5}{6}.$$

S8: Riemann Sums

Let $f(x) = 2x^3$ be defined on the interval $[0, 4]$.

(a) Approximate the area under the curve of the function using four rectangles and right endpoints.

(b) Approximate the area under the curve of the function using four rectangles and left endpoints.

(c) Find a formula for computing R_n, the estimate using n rectangles and right endpoints. (This formula should not have a summation sign or be given as a sum of n terms.)

(d) Use the formula in part (c) to compute the area exactly.

Solution:

(a) $R_4 = 1 \cdot f(1) + 1 \cdot f(2) + 1 \cdot f(3) + 1 \cdot f(4) = 2 + 16 + 54 + 128 = 200$

(b) $L_4 = 1 \cdot f(0) + 1 \cdot f(1) + 1 \cdot f(2) + 1 \cdot f(3) = 0 + 2 + 16 + 54 = 72.$

(c)

$$\begin{aligned} R_n &= \sum_{i=1}^{n} \frac{4}{n} f\left(0 + i\frac{4}{n}\right) = \sum_{i=1}^{n} \frac{4}{n} 2\left((4i/n)\right)^3 \\ &= \sum_{i=1}^{n} \frac{8}{n}\frac{64i^3}{n^3} \\ &= \sum_{i=1}^{n} \frac{512i^3}{n^4} \\ &= \frac{512}{n^4}\sum_{i=1}^{n} i^3 \\ &= \frac{512}{n^4}\frac{n^2(n+1)^2}{4} \\ &= \frac{128n^2(n+1)^2}{n^4} \end{aligned}$$

(d) We can compute

$$\lim_{n\to+\infty} R_n = \lim_{n\to+\infty} \frac{128n^2(n+1)^2}{n^4} = 128.$$