Math 1231 Section 16 Fall 2021
Single-Variable Calculus I Mastery Quiz 2
Due Wednesday, September 22

This week's mastery quiz has three topics. You may attempt all three topics. Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than about 20-30 minutes on this quiz.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in at class/recitation on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Computing Limits
- Secondary Topic 1: Definition of a Limit
- Secondary Topic 2: Squeeze Theorem

Name:

Recitation Section:

Major Topic 1: Computing Limits

1. $\lim_{x \to -2} \frac{x-2}{(x+2)^2} =$

   Solution: The top approaches -4 and the bottom approaches 0, so

   $$\lim_{x \to -2} \frac{x-2}{(x+2)^2} = \pm\infty.$$

   Further, we see that the top is negative and the bottom is always positive, so in fact the limit is $-\infty$.

2. $\lim_{x \to 0} \frac{\sin(5x^2) + \tan^2(x)}{x^2} =$

   Solution:

   $$\begin{aligned} \lim_{x \to 0} \frac{\sin(5x^2) + \tan^2(x)}{x^2} &= \lim_{x \to 0} 5\frac{\sin(5x^2)}{5x^2} + \lim_{x \to 0} \left(\frac{\sin(x)}{\cos(x) \cdot x}\right)^2 \\ &= 5 + 1^2 = 6 \end{aligned}$$

   by the Small Angle Approximation.

3. $\lim_{x \to -\infty} \frac{3x^2 + 2x + 1}{\sqrt{x^4 - x^2 + x}} =$

   Solution:

   $$\begin{aligned} \lim_{x \to -\infty} \frac{3x^2 + 2x + 1}{\sqrt{x^4 - x^2 + x}} &= \lim_{x \to -\infty} \frac{3 + 2/x + 1/x^2}{1/\sqrt{x^4}\sqrt{x^4 - x^2 + x}} \\ &= \lim_{x \to -\infty} \frac{3 + 2/x + 1/x^2}{\sqrt{1 - 1/x^2 + 1/x^3}} \\ &= \frac{3 + 0 + 0}{\sqrt{1 - 0 + 0}} = 3. \end{aligned}$$

Secondary Topic 1: Definition of a Limit

1. Write a formal ϵ-δ proof that $\lim_{x \to 1} 4x - 2 = 2$.

   Solution: Let $\epsilon > 0$ and set $\delta = \epsilon/4$. Then if $0 < |x - 1| < \delta$, we have

   $$|4x - 2 - 2| = |4x - 4| = 4|x - 1| < 4\delta = \epsilon.$$

2. Explicitly naming each limit law you use, compute

   $\lim_{x \to -1} \frac{x^3 + x}{3(x-2)} =$

Solution:

$$\begin{aligned}
\lim_{x\to -1}\frac{x^3+x}{3(x-2)} &= \frac{\lim_{x\to -1} x^3+x}{\lim_{x\to -1} 3(x-2)} && \text{Quotients}\\
&= \frac{\lim_{x\to -1} x^3+x}{\lim_{x\to -1} 3\cdot \lim_{x\to -1} x-2} && \text{Products}\\
&= \frac{\lim_{x\to -1} x^3+\lim_{x\to -1} x}{\lim_{x\to -1} 3\cdot (\lim_{x\to -1} x-\lim_{x\to -1} 2)} && \text{additivity}\\
&= \frac{(\lim_{x\to -1} x)^3+\lim_{x\to -1} x}{\lim_{x\to -1} 3\cdot (\lim_{x\to -1} x-\lim_{x\to -1} 2)} && \text{exponents}\\
&= \frac{(-1)^3+(-1)}{\lim_{x\to -1} 3\,(-1-\lim_{x\to -1} 2)} && \text{identity}\\
&= \frac{-1-1}{3(-1-2)} = \frac{2}{9} && \text{constants.}
\end{aligned}$$

Secondary Topic 2: The Squeeze Theorem

Show that $\lim_{x\to 2}(x-2)\left(1+\sin\left(\frac{1}{x-2}\right)\right) = 0$.

Solution: We know that

$$\begin{aligned}
-1 \le \sin\left(\frac{1}{x-2}\right) \le 1\\
0 \le 1+\sin\left(\frac{1}{x-2}\right) \le 2\\
0 \le (x-2)\left(1+\sin\left(\frac{1}{x-2}\right)\right) \le 2|x-2|.
\end{aligned}$$

Since $\lim_{x\to 2} 0 = 0$ and $\lim_{x\to 2} 2|x-2| = 0$, by the Squeeze Theorem, we know that $\lim_{x\to 2}(x-2)\left(1+\sin\left(\frac{1}{x-2}\right)\right) = 0$.