Math 1231 Section 16 Fall 2021
Single-Variable Calculus I Mastery Quiz 3
Due Monday, September 27

This week's mastery quiz has four topics. **Submit no more than three**. If you already have a 4/4 on M1, do not submit M1 this week; if you have a 2/2 on S1 or S2, do not submit it. (This may mean you only submit S3, and that is perfectly fine.)

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

You shouldn't spend more than about 20-30 minutes on this quiz. Don't worry if you make a minor error, but try to demonstrate that you understand the concepts involved and have mastered the underlying material. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically through Blackboard but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Computing Limits
- Secondary Topic 1: Definition of a Limit
- Secondary Topic 2: Squeeze Theorem
- Secondary Topic 3: Definition of Derivative

Name:

Recitation Section:

Major Topic 1: Computing Limits

(a) $\lim_{x\to 3} \dfrac{\sqrt{7-x}-2}{x-3} =$

Solution:

$$\lim_{x\to 3} \frac{\sqrt{7-x}-2}{x-3} = \lim_{x\to 3} \frac{7-x-4}{(x-3)(\sqrt{7-x}+2)} = \lim_{x\to 3} \frac{-1}{\sqrt{7-x}+2} = \frac{-1}{4}.$$

(b) Compute $\lim_{x\to 0} \dfrac{\sin(3x)\sin(4x)}{x\sin(2x)} =$

Solution:

$$\begin{aligned}\lim_{x\to 0} \frac{\sin(3x)\sin(4x)}{x\sin(2x)} &= \lim_{x\to 0} \frac{\sin(3x)}{3x}\frac{\sin(4x)}{4x}\frac{12x^2}{x\sin(2x)} \\ &= \lim_{x\to 0} \frac{2x}{\sin(2x)}\frac{6x}{x} \\ &= \lim_{x\to 0} \frac{6x}{x} = 6.\end{aligned}$$

(c) $\lim_{x\to 3} \dfrac{1-x}{(x-3)^3} =$

Solution: The top approaches -2 and the bottom approaches 0, so

$$\lim_{x\to 3} \frac{1-x}{(x-3)^3} = \pm\infty.$$

Since the denominator can be either positive or negative, we can't be any more precise than this.

Secondary Topic 1: Definition of a Limit

(a) Write a formal ϵ-δ proof that $\lim_{x\to 3} x+5 = 8$.

Solution: Let $\epsilon > 0$ and set $\delta = \epsilon$. Then if $0 < |x-3| < \delta$, we have

$$|x+5-8| = |x-3| < \delta = \epsilon.$$

(b) Explicitly naming each limit law you use, compute

$\lim_{x\to 0} \dfrac{x^2-x}{2x} =$

Solution:

$$\begin{aligned}
\lim_{x\to 0} \frac{x^2 - x}{2x} &= \lim_{x\to 0} \frac{x-1}{2} && \text{Almost Identical Functions} \\
&= \frac{\lim_{x\to 0} x - 1}{\lim_{x\to 0} 2} && \text{Quotients} \\
&= \frac{\lim_{x\to 0} x - \lim_{x\to 0} 1}{\lim_{x\to 0} 2} && \text{Additivity} \\
&= \frac{\lim_{x\to 0} x - 1}{2} && \text{Constants} \\
&= \frac{-1}{2} && \text{Identity.}
\end{aligned}$$

Secondary Topic 2: The Squeeze Theorem

Show that $\lim_{x\to -1} (x+1)\sin\left(\frac{3}{(x+1)^3}\right) = 0$.

Solution: We know that

$$-1 \le \sin\left(\frac{3}{(x+1)^3}\right) \le 1$$
$$-|x+1| \le (x+1)\sin\left(\frac{3}{(x+1)^3}\right) \le |x+1|.$$

Since $\lim_{x\to -1} -|x+1| = 0$ and $\lim_{x\to -1} |x+1| = 0$, by the Squeeze Theorem, we know that $\lim_{x\to -1}(x+1)\sin\left(\frac{3}{(x+1)^3}\right) = 0$.

Secondary Topic 3: Definition of Derivative

Compute the following derivatives, *directly from the formal definition of derivative.*

1. If $f(x) = x^2 + 2x$, find $f'(2)$.

   Solution:

$$\begin{aligned}
f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\
&= \lim_{h\to 0} \frac{(2+h)^2 + 2(2+h) - 8}{h} \\
&= \lim_{h\to 0} \frac{4 + 4h + h^2 + 4 + 2h - 8}{h} \\
&= \lim_{h\to 0} \frac{h^2 + 6h}{h} \\
&= \lim_{h\to 0} h + 6 = 6.
\end{aligned}$$

2. If $g(x) = \frac{1}{x+2}$, find $g'(x)$.

   Solution:

$$\begin{aligned} g'(x) &= \lim_{h\to 0} \frac{g(x+h)-g(x)}{h} \\ &= \lim_{h\to 0} \frac{\frac{1}{x+h+2} - \frac{1}{x+2}}{h} \\ &= \lim_{h\to 0} \frac{(x+2)-(x+h+2)}{(x+2)(x+h+2)h} \\ &= \lim_{h\to 0} \frac{-h}{(x+2)(x+h+2)h} \\ &= \lim_{h\to 0} \frac{-1}{(x+2)(x+h+2)} = \frac{-1}{(x+2)^2}. \end{aligned}$$