Math 1231 Section 10 Fall 2021
Single-Variable Calculus I Mastery Quiz 4
Due Thursday, October 7

This week's mastery quiz has four topics. You may submit all four (but this policy might not be repeated next time there are four topics). If you already have a 4/4 on M1, do not submit M1 this week; if you have a 2/2 on S3, do not submit it.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

You shouldn't spend more than about 20-30 minutes on this quiz. Don't worry if you make a minor error, but try to demonstrate that you understand the concepts involved and have mastered the underlying material. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically through Blackboard but this should be a last resort.

Topics on This Quiz

- Major Topic 1: Computing Limits
- Major Topic 2: Computing Derivatives
- Major Topic 3: Linear Approximation
- Secondary Topic 3: Definition of Derivative

Name:

Recitation Section:

Major Topic 1: Computing Limits

(a) $\lim_{x\to 3} \dfrac{1}{x-3} + \dfrac{3}{x^2-3x} =$

Solution:

$$\begin{aligned}\lim_{x\to 3} \frac{1}{x-3} + \frac{3}{x^2-3x} &= \lim_{x\to 3} \frac{x^2-3x-3(x-3)}{(x-3)(x^2-3x)} \\ &= \lim_{x\to 3} \frac{x^2-6x+9}{x(x-3)^2} \\ &= \lim_{x\to 3} \frac{1}{x} = \frac{1}{3}.\end{aligned}$$

(b) $\lim_{x\to+\infty} \dfrac{x^2-3x+2}{x^3+x-1} =$

Solution:

$$\lim_{x\to+\infty} \frac{x^2-3x+2}{x^3+x-1} = \lim_{x\to+\infty} \frac{1/x - 3/x^2 + 2/x^3}{1+1/x^2-1/x^3} = \frac{0-0+0}{1+0-0} = 0.$$

(c) $\lim_{x\to 4^+} \dfrac{x+1}{x-4} =$

Solution: The limit of the top is 5 and the limit of the bottom is 0, so the limit is $\pm\infty$. Since the bottom will always be positive as we approach from the right, the overall limit is in fact $+\infty$.

Major Topic 2: Computing Derivatives

Compute the derivative each of the following functions, using any tools we have developed in class.

(a) $(5x^7 - 3x)\left(x^{4/3} + \dfrac{1}{x}\right)$

Solution:

$$(35x^6 - 3)\left(x^{4/3} + \frac{1}{x}\right) + \left(\frac{4}{3}x^{1/3} - \frac{1}{x^2}\right)(5x^7 - 3x).$$

(b) $\dfrac{d}{dx}\sec(\tan(\cos((x+1)^2)))$

Solution:

$$\sec(\tan(\cos((x+1)^2)))\tan(\tan(\cos((x+1)^2)))\sec^2(\cos((x+1)^2))(-\sin((x+1)^2))2(x+1)$$

Major Topic 3: Linear Approximation

(a) Give a formula for a linear approximation of $f(x) = \sqrt{x^3 + 1}$ near the point $a = 2$.

Solution:

$$f'(x) = \frac{1}{2}(x^3 + 1)^{-1/2} 3x^2$$
$$f'(a) = \frac{1}{2}(9)^{-1/2} \cdot 12 = 2$$
$$f(x) = f(a) + f'(a)(x - a) = 3 + 2(x - 2).$$

(b) Use your answer in part (a) to estimate $f(2.1)$.

Solution: $f(2.1) \approx 3 + 2(.1) = 3.2$.

(c) Write the equation for the tangent line to $g(x) = \frac{x+2}{x-5}$ at the point $a = 6$.

Solution:

$$g'(x) = \frac{(x - 5) - (x + 2)}{(x - 5)^2}$$
$$g'(6) = \frac{1 - 8}{1^2} = -7$$
$$y = 8 - 7(x - 6)$$

Secondary Topic 3: Definition of Derivative

Compute the following derivatives, *directly from the formal definition of derivative.*

(a) If $f(x) = 3x^2 - x$, find $f'(-3)$. **Solution:**

$$\begin{aligned} f'(-3) &= \lim_{h \to 0} \frac{f(-3 + h) - f(-3)}{h} \\ &= \lim_{h \to 0} \frac{3(h - 3)^2 - (h - 3) - 30}{h} \\ &= \lim_{h \to 0} \frac{3h^2 - 18h + 27 - h + 3 - 30}{h} \\ &= \lim_{h \to 0} \frac{3h^2 - 19h}{h} \\ &= \lim_{h \to 0} 3h - 19 = -19. \end{aligned}$$

(b) If $g(x) = \frac{x}{x+2}$, find $g'(x)$. **Solution:**

$$\begin{aligned}
g'(x) &= \lim_{h\to 0} \frac{g(x+h) - g(x)}{h} \\
&= \lim_{h\to 0} \frac{\frac{x+h}{x+h+2} - \frac{x}{x+2}}{h} \\
&= \lim_{h\to 0} \frac{(x+h)(x+2) - x(x+h+2)}{h(x+2)(x+h+2)} \\
&= \lim_{h\to 0} \frac{x^2 + hx + 2x + 2h - x^2 - xh - 2x}{h(x+2)(x+h+2)} \\
&= \lim_{h\to 0} \frac{2h}{h(x+2)(x+h+2)} \\
&= \lim_{h\to 0} \frac{2}{(x+2)(x+h+2)} = \frac{2}{(x+2)^2}.
\end{aligned}$$