Math 1231 Section 10 Fall 2021
Single-Variable Calculus I Mastery Quiz 6
Due Thursday, October 28

This week's mastery quiz has five topics. **You may submit up to four**.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

You shouldn't spend more than about 20-30 minutes on this quiz. Don't worry if you make a minor error, but try to demonstrate that you understand the concepts involved and have mastered the underlying material. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically through Blackboard but this should be a last resort.

Topics on This Quiz

- Major Topic 2: Computing Derivatives
- Major Topic 3: Linear Approximation
- Major Topic 4: Extrema and Optimization
- Secondary Topic 4: Rates of Change
- Secondary Topic 5: Related Rates

Name:

Recitation Section:

Major Topic 2: Computing Derivatives

Compute the derivative each of the following functions, using any tools we have developed in class.

(a) Find a formula for y' in terms of x and y if $xy^3 = \sqrt{x^2+y^2}$.

Solution: Using implicit differentiation, we have

$$\begin{aligned} y^3 + 3xy^2y' &= \frac{2x+2yy'}{2\sqrt{x^2+y^2}} \\ &= \frac{x+yy'}{\sqrt{x^2+y^2}} \\ y^3 - \frac{x}{\sqrt{x^2+y^2}} &= \frac{yy'}{\sqrt{x^2+y^2}} - 3xy^2y' \\ y' &= \frac{y^3 - \frac{x}{\sqrt{x^2+y^2}}}{\frac{y}{\sqrt{x^2+y^2}} - 3xy^2} \\ &= \frac{y^3\sqrt{x^2+y^2} - x}{y - 3xy^2\sqrt{x^2+y^2}}. \end{aligned}$$

(b) Compute $\dfrac{d}{dx}\sec\left(\dfrac{x^3-x}{\sqrt[5]{x}+1}\right) =$

Solution:

$$\sec\left(\frac{x^3-x}{\sqrt[5]{x}+1}\right)\tan\left(\frac{x^3-x}{\sqrt[5]{x}+1}\right)\frac{(3x^2-1)(\sqrt[5]{x}+1) - \frac{1}{5}x^{-4/5}(x^3-x)}{(\sqrt[5]{x}+1)^2}$$

Major Topic 3: Linear Approximation

(a) Write a tangent line to the curve $x^2y^2 = 5+x+y$ at the point $(1,3)$.

Solution: Implicit differentiation gives us

$$\begin{aligned} 2xy^2 + 2x^2yy' &= 1+y' \\ 2\cdot 1\cdot 9 + 2\cdot 1^2\cdot 3\cdot y' &= 1+y' \\ 18+6y' &= 1+y' \\ 5y' &= -17 \\ y' &= -17/5 \end{aligned}$$

and thus the tangent line has equation

$$y-3 = \frac{-17}{5}(x-1).$$

Alternatively we can compute

$$\begin{aligned} 2xy^2 + 2x^2yy' &= 1 + y' \\ y'(2x^2y - 1) &= 1 - 2xy^2 \\ y' &= \frac{1 - 2xy^2}{2x^2y - 1} \end{aligned}$$

(b) Use linear approximation to estimate $\sqrt[4]{14}$.

Solution: We take $f(x) = \sqrt[4]{x}$, and take $a = 16$. Then

$$\begin{aligned} f'(x) &= \frac{1}{4}x^{-3/4} \\ f'(16) &= \frac{1}{4}(16)^{-3/4} = \frac{1}{4 \cdot 8} = \frac{1}{32} \\ f(x) &\approx f(a) + f'(a)(x - a) = 2 + \frac{1}{32}(14 - 16) = 2 - \frac{1}{16} = \frac{31}{16}. \end{aligned}$$

M4: Extrema and Optimization

(a) Find the absolute extrema of $g(x) = 3x^4 - 2x^3 - 3x^2 + 5$ on the interval $[-1, 2]$, and justify your claim that these are the absolute extrema.

Solution:

g is continuous on the closed interval $[-1, 2]$, so by the Extreme Value Theorem it has a maximum and a minimum on the interval. This must happen at a critical point or an endpoint. (This argument is necessary! Otherwise there's no reason to expect the largest local max to be a global max.)

We compute

$$g'(x) = 12x^3 - 6x^2 - 6x = 6x(2x^2 - x - 1) = 6x(2x + 1)(x - 1)$$

which is zero at $0, 1, -1/2$. Then we compute

$$\begin{aligned} g(-1) &= 7 \\ g(-1/2) &= \frac{3}{16} + \frac{1}{4} - \frac{3}{4} + 5 = 5 - \frac{5}{16} = \frac{75}{16} = 4.6875 \\ g(0) &= 5 \\ g(1) &= 3 \\ g(2) &= 48 - 16 - 12 + 5 = 25. \end{aligned}$$

Thus g has an absolute maximum of 25 at 2, and an absolute minimum of 3 at 1.

(b) Find all the critical points of $f(x) = \sqrt[3]{x^3 - 3x}$.

Solution:

We have

$$f'(x) = \frac{1}{3}(x^3 - 3x)^{-2/3}(3x^2 - 3) = \frac{(x-1)(x+1)}{\sqrt[3]{x(x^2-3)}^2}.$$

This is zero when $x = \pm 1$ and is undefined when $x = 0$ or $x = \pm\sqrt{3}$. Thus the critical points are $-\sqrt{3}, -1, 0, 1, \sqrt{3}$.

Secondary Topic 4: Rates of Change

(a) Suppose that a factory produces widgets, and if p people work at the factory then they will produce a total of $W(p) = 30\sqrt{p}$ widgets.

(i) What does the derivative $W'(p)$ represent, and what are its units?
Solution: The derivative is the rate at which the number of widgets increases as we add more people to the factory (called the marginal product of labor). Its units are widgets per person.

(ii) Calculate $W'(9)$. What does this represent in the real world?
Solution: $W'(p) = \frac{15}{\sqrt{p}}$ so $W'(9) = 5$. So moving from nine people to ten people working at the factory will lead to the production of five extra widgets.

(b) Suppose the distance between two particles in centimeters is given as a function of time in seconds by the formula $d(t) = t^3 + 4t^2 + 5t + 4$.

(i) When is the velocity zero?
Solution: $d'(t) = 3t^2 + 8t + 5 = (3t+5)(t+1)$ so the velocity is zero when $t = -1, -5$.

(ii) When is the acceleration zero?
Solution: $d''(t) = 6t + 8$ is zero when $t = -4/3$.

S5: Related Rates

A rocket is taking off with a perfectly vertical path, and is being tracked by a radar station on the ground four miles from the launch pad. How fast is the rocket rising when it is three miles high and its distance from the radar station is increasing at a rate of 3000 miles per hour.

Solution:

We know one speed and want to know another, and we also know distances. This means we probably want to use the distance formula and take its derivative to find speeds.

We write $h = 3$ miles, and can work out that $d = 5$ miles. We know that $d' =$ miles per hours.

We know that $d^2 = h^2 + 4^2$ and thus $2dd' = 2hh'$. Plugging in values gives us

$$2 \cdot 5\text{mi} \cdot 3000\text{mi/hr} = 2 \cdot 3\text{mi} \cdot h'$$
$$h' = 5000\text{mi/hr}.$$

Thus the rocket is rising at 5000 miles per hour.