Math 1231 Section 16 Fall 2021
Single-Variable Calculus I Mastery Quiz65
Due Monday, October 25

This week's mastery quiz has four topics. **Submit no more than three**. You may have completed M2 or S4 already, based on your quiz scores. I will try to get the mastery scores from the midterm updated over the weekend, but I don't know when that will happen (and they definitely won't all be uploaded.)

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

You shouldn't spend more than about 20-30 minutes on this quiz. Don't worry if you make a minor error, but try to demonstrate that you understand the concepts involved and have mastered the underlying material. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically through Blackboard but this should be a last resort.

Topics on This Quiz

- Major Topic 2: Computing Derivatives
- Major Topic 3: Linear Approximation
- Secondary Topic 4: Rates of Change
- Secondary Topic 5: Related Rates

Name:

Recitation Section:

Major Topic 2: Computing Derivatives

(a) Find a formula for y' in terms of x and y if $x^3y + x^2y^2 + y^4 = 0$.

Solution:

$$\begin{aligned} 3x^2y + x^3y' + 2xy^2 + 2x^2yy' + 4y^3y' &= 0 \\ x^3y' + 2x^2yy' + 4y^3y' &= -3x^2y - 2xy^2 \\ y' &= -\frac{3x^2y + 2xy^2}{x^3 + 2x^2y + 4y^3}. \end{aligned}$$

(b) Compute $\dfrac{d}{dx}g(x) = \left(\dfrac{x\csc(x)}{\sqrt{x^3 - x}}\right)^3$

Solution:

$$g'(x) = 3\left(\frac{x\csc(x)}{\sqrt{x^3 - x}}\right)^2 \frac{\big(\csc(x) - x\csc(x)\cot(x)\big)\sqrt{x^3 - 1} - x\csc(x)\frac{1}{2}(x^3 - x)^{-1/2}(3x^2 - 1)}{x^3 - x}.$$

Major Topic 3: Linear Approximation

(a) Estimate $\sqrt[4]{15}$ using a linear approximation of the function $\sqrt[4]{x}$ at the point 16.

Solution: We have $h(x) = \sqrt[4]{x}$ and so $h'(x) = \frac{1}{4}x^{-3/4} = \frac{1}{4\sqrt[4]{x^3}}$. Thus in particular, we have $h'(16) = \frac{1}{4\sqrt[4]{16^3}} = \frac{1}{4\cdot 2^3} = 1/32$.

The tangent line approximation is

$$y - 2 = \frac{1}{32}(x - 16)$$

so we have

$$\begin{aligned} f(x) &\approx \frac{1}{32}(x - 16) + 2 \\ f(15) &\approx \frac{1}{32}(-1) + 2 = 2 - \frac{1}{32} = \frac{63}{32}. \end{aligned}$$

(b) Find an equation of the line tangent to $y = \frac{x+1}{x-1}$ at the point $x = 2$.

Solution:

$$y' = \frac{(x-1) - (x+1)}{(x-1)^2}$$

and thus at $x = 2$ we have $y' = \frac{1-3}{1^2} = -2$. The point on the curve is $(2, 3)$, so we have the equation

$$y - 3 = -2(x - 2) \qquad \text{or} \qquad y = 7 - 2x.$$

Secondary Topic 4: Rates of Change

(a) The force a magnet exerts on a piece of iron depends on the distance between the magnet and the metal. Let $F(d) = \frac{2}{d^2}$ give the force exerted by the magnet in Newtons, where d is the distance between them in meters.

(i) What does the derivative $F'(d)$ represent, and what are its units?

Solution: The derivative is the rate at which the amount of force changes as you change the distance between the magnet and the iron; its units are Netwons per meter.

(ii) Calculate $F'(2)$. What does this tell you physically?

Solution: $F'(d) = \frac{-4}{d^3}$ so $F'(3) = \frac{-4}{8} = -1/2$. This means that moving the iron another meter away from the magnet should reduce the force by about half a Newton.

(b) Suppose the distance between two particles in centimeters is given as a function of time in seconds by the formula $d(t) = t + \frac{1}{t}$.

(i) When is the velocity zero?

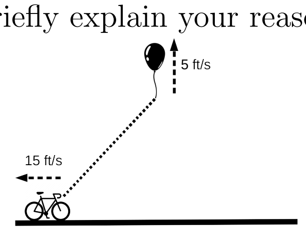
Solution: $d'(t) = 1 - 1/t^2$ so the velocity is zero when $t = \pm 1$.

(ii) When is the acceleration zero?

Solution: $d''(t) = 2/t^3$ is never zero.

Secondary Topic 5: Related Rates

A balloon is rising at a constant speed of 5 feet per second. A boy is cycling along a straight road at a speed of 15 feet per second. When he passes under the balloon, it is 45 feet above him. How fast is the distance between the boy and the balloon increasing 3 seconds later? Briefly explain your reasoning.



Solution:

We see that the height of the balloon is $h = 60$ feet, and the derivative is $h' = 5$ feet per second. The distance between the boy and the point under the balloon is $w = 45$ feet and the derivative is $w' = 15$ feet per second. The distance between them is given by $d^2 = w^2 + h^2$, and so we can compute first that the current distance is 75 feet, and then that

$$\begin{aligned} 2dd' &= 2ww' + 2hh' \\ dd' &= ww' + hh' \\ 75d' &= 45 \cdot 15 + 60 \cdot 5 = 675 + 300 = 975 \\ d' &= \frac{975}{75} = \frac{325}{25} = 13. \end{aligned}$$

Thus the distance is increasing by 13 feet per second.