Math 1231 Section 10 Fall 2021
Single-Variable Calculus I Mastery Quiz 7
Due Thursday, November 4

This week's mastery quiz has four topics. **You may submit up to three**. This is the last week for topics M2 and S5. M3 is not on this week's quiz, but will reappear on next week's quiz.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

You shouldn't spend more than about 20-30 minutes on this quiz. Don't worry if you make a minor error, but try to demonstrate that you understand the concepts involved and have mastered the underlying material. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically through Blackboard but this should be a last resort.

Topics on This Quiz

- Major Topic 2: Computing Derivatives
- Major Topic 4: Extrema and Optimization
- Secondary Topic 5: Related Rates
- Secondary Topic 6: Curve Sketching

Name:

Recitation Section:

Major Topic 2: Computing Derivatives

Compute the derivative each of the following functions, using any tools we have developed in class.

(a) Compute $\frac{d}{dx}\cos^2\left(\tan^2\left(\sec^2(\sqrt{x}+x)\right)\right)$.

Solution:

$$\begin{aligned}2\cos\left(\tan^2\left(\sec^2(\sqrt{x}+x)\right)\right)\left(-\sin\left(\tan^2\left(\sec^2(\sqrt{x}+x)\right)\right)\right)\\ \cdot 2\tan\left(\sec^2(\sqrt{x}+x)\right)\sec^2\left(\sec^2(\sqrt{x}+x)\right)\\ \cdot 2\sec(\sqrt{x}+x)\sec(\sqrt{x}+x)\tan(\sqrt{x}+x)\left(\frac{1}{2x}+1\right)\end{aligned}$$

(b) Find a formula for $\frac{d^2y}{dx^2}$ in terms of x and y if $\sin(xy) = x + y$.

Solution: Using implicit differentiation, we have

$$\begin{aligned}\cos(xy)(1+xy') &= 1+y'\\ \cos(xy)-1 &= (1-x)y'\\ y' &= \frac{\cos(xy)-1}{1-x}\\ y'' &= \frac{-\sin(xy)(1+xy')(1-x)-(\cos(xy)-1)(-1)}{(1-x)^2}\\ &= \frac{-\sin(xy)\left(1+x\frac{\cos(xy)-1)}{1-x}\right)(1-x)-(\cos(xy)-1)(-1)}{(1-x)^2}\end{aligned}$$

M4: Extrema and Optimization

(a) Classify the critical points and relative extrema of $g(x) = \frac{2x-1}{x^2+2}$.

Solution: We have

$$\begin{aligned}g'(x) &= \frac{2(x^2+2)-2x(2x-1)}{(x^2+2)^2} = \frac{-2x^2+2x+4}{(x^2+2)^2}\\ &= -2\frac{x^2-x-2}{(x^2+2)^2} = -2\frac{(x-2)(x+1)}{(x^2+2)^2}\end{aligned}$$

so the critical points are 2 and −1. (The derivative is defined everywhere).

To classify these critical points we need to use either the first or second derivative test. I think the first derivative test looks easier here, purely because I don't want to compute the second derivative. I get the table

	$x-2$	$x+1$	$\frac{-2}{(x^2+2)^2}$	$g'(x)$
$x < -1$	$-$	$-$	$-$	$-$
$-1 < x < 2$	$-$	$+$	$-$	$+$
$2 < x$	$+$	$+$	$-$	$-$

Thus we see that there is a relative minimum at -1 and a relative maximum at 2.

But we could use the second derivative test if we really wanted to. We compute

$$g''(x) = -2\frac{(2x-1)(x^2+2)^2 - 2(x^2+2)2x(x^2-x-2)}{(x^2+2)^4}$$

$$g''(-1) = -2\frac{(-3)(3)^2 - 2(3)(-2)(0)}{3^4} = \frac{-2\cdot(-27)}{3^4} = 2/3 > 0$$

$$g''(2) = -2\frac{3(6)^2 - 2(6)4(0)}{6^4} = \frac{-1}{6} < 0.$$

Thus $g''(-1) > 0$ so g has a minimum at -1; and $g''(2) < 0$ so g has a maximum at 2.

(b) Find the absolute extrema of $g(x) = x^3 - 3x^2 - 9x + 5$ on $[-2, 4]$, and justify your claim that these are in fact absolute extrema.

Solution: g is continuous on the closed interval $[-2, 4]$ so by the extreme value theorem it has an absolute maximum and an absolute minimum.

We compute $g'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1)$ is always defined, and is zero if $x = -1$ or $x = 3$. So the critical points are -1 and 3, and we need to check the points $-2, -1, 3, 4$.

$$g(-2) = 3 \qquad g(-1) = 10$$
$$g(3) = -22 \qquad g(4) = -15.$$

Thus g has a maximum of 10 at -1 and a minimum of -22 at 3.

S5: Related Rates

A snowball is melting such that its surface area is decreasing at $1\text{cm}^2/\text{min}$. When the radius is 8cm, how quickly is the radius decreasing? Please **write a complete sentence** to answer this question at the end of your work.

(The surface area of a sphere of radius r is $4\pi r^2$.)

Solution: We have $S = 4\pi r^2$, so $S' = 8\pi r r'$. When the radius is 8cm we have

$$S' = 8\pi \cdot 8\text{cm} \cdot r'$$
$$-1\text{cm}^2/\text{min} = 64\pi r'$$
$$r' = \frac{-1}{64\pi}\text{cm/min}.$$

Thus when the radius is 8cm, the radius of the snowball is decreasing by $\frac{1}{64\pi}$ centimeters per minute.

S6: Curve Sketching

Let $f(x) = \dfrac{(x-2)^2}{x-1}$. We can compute that

$$f'(x) = \frac{x(x-2)}{(x-1)^2}$$
$$f''(x) = \frac{2}{(x-1)^3}.$$

Sketch a graph of f. Your answer should discuss the domain, roots, asymptotes, limits at infinity, critical points and values, intervals of increase and decrease, and concavity and points of inflection.

Solution:

The function is defined for all real numbers except 1. We compute that $\lim_{x\to 1^+} f(x) = +\infty$ and $\lim_{x\to 1^-} f(x) = -\infty$. There is a root of f at $x = 2$, and we compute that $\lim_{x\to+\infty} f(x) = +\infty$ and $\lim_{x\to-\infty} f(x) = -\infty$.

$f'(x)$ is undefined at $x = 1$ and is 0 at $0, 2$ so the critical points are $0, 1, 2$. We compute that $f(0) = -4$ and $f(2) = 0$. We make a chart:

	x	$x-2$	$(x-1)^{-2}$	$f'(x)$
$x < 0$	$-$	$-$	$+$	$+$
$0 < x < 1$	$+$	$-$	$+$	$-$
$1 < x < 2$	$+$	$-$	$+$	$-$
$2 < x$	$+$	$+$	$+$	$+$

Thus f is increasing on $(-\infty, 0)$ and $(2, +\infty)$, and is decreasing on $(0, 2)$. It has relative a relative maximum at $(0, -4)$ and a relative minimum at $(2, 0)$; it doens't have a value at 1.

$f''(x)$ is undefined at 1 and is never 0, so the only possible point of inflection is 1. We see that $f''(x)$ is negative if $x < 1$ and positive if $x > 1$, so f is concave down on $(-\infty, 1)$ and concave up on $(1, +\infty)$.