Math 1231 Section 16 Fall 2021
Single-Variable Calculus I Mastery Quiz 7
Due Monday, November 1

This week's mastery quiz has four topics. **Submit no more than three**.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

You shouldn't spend more than about 20-30 minutes on this quiz. Don't worry if you make a minor error, but try to demonstrate that you understand the concepts involved and have mastered the underlying material. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically through Blackboard but this should be a last resort.

Topics on This Quiz

- Major Topic 2: Computing Derivatives
- Major Topic 3: Linear Approximation
- Major Topic 4: Optimization
- Secondary Topic 5: Related Rates

Name:

Recitation Section:

Major Topic 2: Computing Derivatives

(a) Compute $\dfrac{d}{dx}\csc^{7/3}\left(\dfrac{x^2+\sin(x)}{\sqrt{x}-\cot(x)}\right)$.

Solution:

$$\frac{7}{3}\csc^{4/3}\left(\frac{x^2+\sin(x)}{\sqrt{x}-\cot(x)}\right)\left(-\csc\left(\frac{x^2+\sin(x)}{\sqrt{x}-\cot(x)}\right)\cot\left(\frac{x^2+\sin(x)}{\sqrt{x}-\cot(x)}\right)\right)$$
$$\cdot\frac{(2x+\cos(x))(\sqrt{x}-\cot(x))-\left(\frac{1}{2}x^{-1/2}+\csc^2(x)\right)(x^2+\sin(x))}{(\sqrt{x}-\cot(x))^2}.$$

(b) Find a formula for $\frac{d^2y}{dx^2}$ if $x^3 = xy+1$.

Solution:

$$\begin{aligned}
3x^2 &= y + xy' \\
y' &= \frac{3x^2-y}{x} && = 3x - \frac{y}{x} \\
y'' &= \frac{(6x-y')x-(3x^2-y)}{x^2} && = 3 - \frac{y'x-y}{x^2} \\
&= \frac{\left(6x-\frac{3x^2-y}{x}\right)x-(3x^2-y)}{x^2} && = 3-\frac{\frac{y}{x}\cdot x - y}{x^2} = 3.
\end{aligned}$$

Major Topic 3: Linear Approximation

(a) Find an equation for the line tangent to the curve $x^2y - xy^3 = xy + 3$ at the point $(3,1)$.

Solution:

$$\begin{aligned}
2xy + x^2y' - y^3 - 3xy^2y' &= y + xy' \\
6 + 9y' - 1 - 9y' &= 1 + 3y' \\
4 &= 3y' \\
y' &= 4/3
\end{aligned}$$

and thus an equation for the tangent line is

$$y - 1 = \frac{4}{3}(x-3).$$

(b) Find a linear approximation to the function $f(x) = \frac{x^3}{1+x}$ near the point $a = 1$ and use it to approximate $f(1.3)$.

Solution:

$$
\begin{aligned}
f(1) &= \frac{1}{2} \\
f'(x) &= \frac{3x^2(1+x) - x^3}{(1+x)^2} \\
f'(1) &= \frac{6-1}{4} = \frac{5}{4} \\
f(x) &\approx \frac{1}{2} + \frac{5}{4}(x-1) \\
f(1.3) &\approx \frac{1}{2} + \frac{5}{4} \cdot .3 = \frac{20}{40} + \frac{15}{40} = \frac{35}{40} = \frac{7}{8}.
\end{aligned}
$$

Major Topic 4: Optimization

(a) Classify the critical points and relative extrema of $f(x) = 5 + 8x^3 + x^4$.

Solution:

We have $f'(x) = 24x^2 + 4x^3 = 4x^2(6+x)$. So the critical points are $-6, 0$. We could try the second derivative test: we get $f''(x) = 48x + 12x^2$. Then $f''(-6) = 12(-24+36) = 144 > 0$, so this is a relative minimum. But $f''(0) = 0$ which doesn't tell us anything. We have to pass to the first derivative test.

We make a chart:

	$4x^2$	$6+x$	$f'(x)$
$x < -6$	$+$	$-$	$-$
$-6 < x < 0$	$+$	$+$	$+$
$0 < x$	$+$	$+$	$+$

Thus f is decreasing when $x < -6$ and increasing when $x > -6$, and so we see that f has a minimum at $x = -6$, and a point which is neither a maximum nor a minimum at $x = 0$,.

(b) Find all the critical points of $g(x) = \frac{x^2-3x-4}{x+5}$

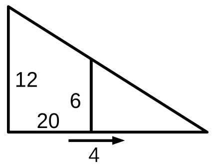
Solution:

$$
\begin{aligned}
g'(x) &= \frac{(2x-3)(x+5) - (x^2-3x-4)}{(x+5)^2} \\
&= \frac{x^2+10x-11}{(x+5)^2} \\
&= \frac{(x+11)(x-1)}{(x+5)^2}.
\end{aligned}
$$

Thus $g'(x) = 0$ when $x = -11$ or $x = 1$, and $g'(x)$ is undefined when $x = -5$. So the critical points are $1, -5, -11$.

Secondary Topic 5: Related Rates

A street light is mounted at the top of a 12-foot-tall pole. A six-foot-tall man walks straight away from the pole at 4 feet per second. How fast is his shadow growing longer when he is twenty feet from the pole?



Solution: Let d be the distance of the man from the pole. Then $d = 20$ and $d' = 4$. If s is the length of the shadow, then we have $s/6 = (d+s)/12$ so we get

$$\begin{aligned} s &= \frac{d+s}{2} \\ s' &= d'/2 + s'/2 \\ s'/2 &= d'/2 \\ s' &= d' = 4. \end{aligned}$$

Thus the length of the shadow is growing at 4 feet per second.