Math 1231 Section 10 Fall 2021
Single-Variable Calculus I Mastery Quiz 8
Due Thursday, November 11

This week's mastery quiz has four topics. **You may submit up to three**. This is the last week for topics M3 and S6.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

You shouldn't spend more than about 20-30 minutes on this quiz. Don't worry if you make a minor error, but try to demonstrate that you understand the concepts involved and have mastered the underlying material. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically through Blackboard but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Linear Approximation
- Major Topic 4: Extrema and Optimization
- Secondary Topic 6: Curve Sketching
- Secondary Topic 7: Approximation

Name:

Recitation Section:

Major Topic 3: Linear Approximation

(a) Use linear approximation to estimate $f(9.1)$ if $f(x) = 3\sqrt{x} + 2x$.

Solution:

$$\begin{aligned} f'(x) &= \frac{3}{2\sqrt{x}} + 2 \\ f'(9) &= \frac{3}{6} + 2 = 5/2 \\ f(9) &= 27 \\ f(x) &\approx 27 + \frac{5}{2}(x-9) \\ f(9.1) &\approx 27 + \frac{5}{2}(.1) = 27.25. \end{aligned}$$

(b) Find a line tangent to the graph of $g(x) = \tan(x)\sec(x)$ at the point $x = \pi/6$.

Solution: We have

$$\begin{aligned} g(\pi/6) &= \tan(\pi/6)\sec(\pi/6) = \frac{\sin(\pi/6)}{\cos(\pi/6)^2} = \frac{1/2}{3/4} = \frac{2}{3} \\ g'(x) &= \sec^2(x)\sec(x) + \tan(x)\sec(x)\tan(x) = \sec^3(x) + \sec(x)\tan^2(x) \\ g'(\pi/6) &= \frac{8}{3\sqrt{3}} + \frac{2}{3\sqrt{3}} = \frac{10}{3\sqrt{3}} \\ g(x) &\approx \frac{2}{3} + \frac{10}{3\sqrt{3}}(x - \pi/6). \end{aligned}$$

M4: Extrema and Optimization

(a) Suppose that a company that produces and sells x units of a product makes a revenue of $R(x) = 260x - 9x^2/10$ and has costs given by $C(x) = 1000 + 100x + x^2/10$. What is the maximum profit that can be made (where profit is revenues minus costs)?

Solution: Our profit function is $P(x) = R(x) - C(x) = -1000 + 160x - x^2$. Then

$$\begin{aligned} P'(x)&160 - 2x \\ 160 &= 2x \\ 80 &= x \end{aligned}$$

We can check that this is truly a maximum by the second derivative: $P''(x) = -2 < 0$ so we have a local maximum.

Or we can see that $P'(x) > 0$ when $x < 80$ and $P'(x) < 0$ when $x > 80$, so the function is increasing until 80 and decreasing after.

The profit at this quantity is

$$P(80) = -1000 + 160(80) - (80)^2 = -1000 + 12800 - 6400 = 5400.$$

(b) Classify the critical points and relative extrema of $h(x) = \sin(x) + \cos(x)$ on $[0, 2\pi]$.

Solution: We have

$$h'(x) = \cos(x) - \sin(x)$$

so $h'(x)$ is defined everywhere, and has critical points where $\cos(x) = \sin(x)$. This happens when $x = \pi/4, 5\pi/4, 9\pi/4, \cdots = \pi/4 + n\pi$. We only need to care about $\pi/4$ and $5\pi/4$.

We can classify these points in two ways. We can use the first derivative test or the second derivative test. In these solutions I'll do both.

For the second derivative test we compute:

$$\begin{aligned} h''(x) &= -\sin(x) - \cos(x) \\ h''(\pi/4) &= -\sqrt{2}/2 - \sqrt{2}/2 = -\sqrt{2} < 0 \\ h''(5\pi/4) &= \sqrt{2}/2 + \sqrt{2}/2 = \sqrt{2} > 0. \end{aligned}$$

Thus h has a local maximum at $\pi/4$ and has a local minimum at $5\pi/4$.

For the first derivative test we make a chart:

	$h'(x)$
$0 < x < \pi/4$	$+$
$\pi/4 < x < 5\pi/4$	$-$
$5\pi/4 < x < 2\pi$	$+$

so h has a relative maximum at $\pi/4$ and a relative minimum at $5\pi/4$.

S6: Curve Sketching

Sketch the graph of $g(x) = 3x^4 - 4x^3 - 36x^2 + 64 = (x+2)^2(3x-4)(x-4)$ have $g'(x) = 12x^3 - 12x^2 - 72x = 12x(x-3)(x+2)$ and $g''(x) = 36x^2 - 24x - 72 = 12(3x^2 - 2x - 6)$.

You should discuss the domain, limits, critical points, intervals of increase and decrease, concavity, and possible points of inflection.

Solution:

(i) The domain is all reals.

(ii) The roots are at $x = -2, 4/3, 4$.

(iii) We have $\lim_{x\to+\infty} g(x) = \lim_{-\infty} g(x) = +\infty$.

(iv)

$$g'(x) = 12x^3 - 12x^2 - 72x = 12x(x^2 - x - 6) = 12x(x-3)(x+2).$$

This is defined everywhere, and has roots at $-2, 0, 3$.

Thus the critical points are $x = -2, 0, 3$. We compute $g(-2) = 0, g(0) = 64, g(3) = 5^2 \cdot 5 \cdot (-1) = -125$.

(v) We make a chart:

	$12x$	$(x-3)$	$(x+2)$	$g'(x)$
$x < -2$	$-$	$-$	$-$	$-$
$-2 < x < 0$	$-$	$-$	$+$	$+$
$0 < x < 3$	$+$	$-$	$+$	$-$
$3 < x$	$+$	$+$	$+$	$+$

This implies relative minima at -2 and at 3, and a relative maximum at 0.

(vi) We compute

$$g''(x) = 36x^2 - 24x - 72 = 12(3x^2 - 2x - 6)$$

which has roots

$$x = \frac{2 \pm \sqrt{4+72}}{6} = \frac{1 \pm \sqrt{19}}{3}.$$

and there are critical points at roughly -1 and $5/3$.

Plugging in some values we have

$$g''(2) = 12 \cdot (2) = 24 > 0$$
$$g''(0) = -72 < 0$$
$$g''(-2) = 12 \cdot 10 = 120 > 0.$$

Thus the function is concave up on $\left(-\infty, \frac{1-\sqrt{19}}{3}\right) \cup \left(\frac{1+\sqrt{19}}{3}, +\infty\right)$ and is concave down on $\left(\frac{1-\sqrt{19}}{3}, \frac{1+\sqrt{19}}{3}\right)$.

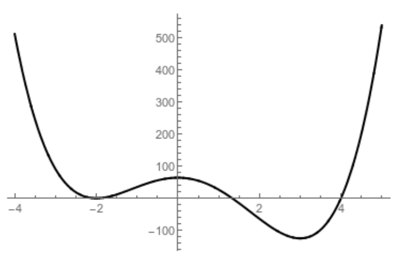

Figure 1: Graph of $g(x)$

S7: Approximation

I apparently posted two different versions of this question at different times; it's fine to have answered either one. I've included boht sets of solutions below.

(a) Find a formula for the quadratic approximation of $f(x) = \sqrt{3x+1}$ near the point $a = 1$, and use it to estimate $f(1.01)$.

Solution: We compute

$$f(1) = 2$$
$$f'(x) = \frac{3}{2}(3x+1)^{-1/2} \qquad f'(1) = \frac{3}{4}$$
$$f''(x) = \frac{-9}{4}(3x+1)^{-3/2} \qquad f''(1) = \frac{-9}{32}$$

and thus we have

$$f(x) \approx 2 + \frac{3}{4}(x-1) - \frac{9}{64}(x-1)^2$$
$$f(1.01) \approx 2 + \frac{.03}{4} - \frac{.09}{64} = \frac{12839}{6400} = 2.00609375.$$

(b) Use two steps of Newton's method to estimate $\sqrt{8}$ starting from $x_0 = 3$. (You should compute x_2.)

Solution: We're looking for a root of $g(x) = x^2 - 8$. So we compute $g'(x) = 2x$, and we get

$$x_1 = 3 - \frac{g(3)}{g'(3)} = 3 - \frac{1}{6} = \frac{17}{6}$$
$$x_2 = \frac{17}{6} - \frac{g(17/6)}{g'(17/6)} = \frac{17}{6} - \frac{289/36 - 288/36}{17/3} = \frac{17}{6} - \frac{1}{12 \cdot 17} = \frac{577}{204} \approx 2.8284$$

(The true answer is approximately 2.8284, so yay for us.)

or

(a) Find a formula for the quadratic approximation of $f(x) = \sin(x^2 + x)$ near the point $a = 0$, and use it to estimate $f(.1)$.

Solution: We compute

$$f(0) = 0$$
$$f'(x) = \cos(x^2+x)(2x+1) \qquad f'(0) = 1$$
$$f''(x) = -\sin(x^2+x)(2x+1)^2 + 2\cos(x^2+x) \qquad f''(0) = 2$$

and thus we have

$$f(x) \approx 0 + 1(x) + \frac{2}{2}(x)^2 = x + x^2$$
$$f(.1) \approx .1 + .01 = .11.$$

(b) Use two steps of Newton's method to estimate a solution to $x^3 + x = 1$ starting from $x_0 = 1$. (You should compute x_2.)

Solution: We're looking for a root of $g(x) = x^3+x-1$. So we compute $g'(x) = 3x^2+1$, and we get

$$x_1 = 1 - \frac{g(1)}{g'(1)} = 1 - \frac{1}{4} = \frac{3}{4}$$

$$x_2 = \frac{3}{4} - \frac{g(3/4)}{g'(3/4)} = \frac{3}{4} - \frac{27/64 + 3/4 - 1}{27/16 + 1} = \frac{3}{4} - \frac{11/64}{43/16} = \frac{3}{4} - \frac{11}{172} = \frac{118}{172} \approx .686.$$

(The true answer is approximately .68233, so yay for us.)