Math 1231 Section 16 Fall 2021
Single-Variable Calculus I Mastery Quiz 8
Due Monday, November 8

This week's mastery quiz has three topics. You may submit all three.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

You shouldn't spend more than about 20-30 minutes on this quiz. Don't worry if you make a minor error, but try to demonstrate that you understand the concepts involved and have mastered the underlying material. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically through Blackboard but this should be a last resort.

Topics on This Quiz

- Major Topic 3: Linear Approximation
- Major Topic 4: Optimization
- Secondary Topic 6: Curve Sketching

Name:

Recitation Section:

Major Topic 3: Linear Approximation

(a) Find an equation for the line tangent to the curve $3x^2y + 5xy^2 = 2x$ at the point $(1, -1)$.

Solution:

$$\begin{aligned} 6xy + 3x^2y' + 5y^2 + 10xyy' &= 2 \\ -6 + 3y' + 5 - 10y' &= 2 \\ -7y' &= 3 \\ y' &= -3/7 \end{aligned}$$

and thus an equation for the tangent line is

$$y + 1 = \frac{-3}{7}(x - 1).$$

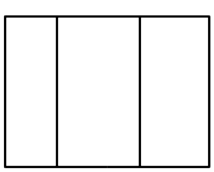
(b) Find a linear approximation to the function $f(x) = \sin(x)\cos(x)$ near the point $a = \pi/3$ and use it to approximate $f(\pi/2)$.

Solution:

$$\begin{aligned} f(\pi/3) &= \frac{\sqrt{3}}{4} \\ f'(x) &= \cos^2(x) - \sin^2(x) \\ f'(\pi/3) &= 1/4 - 3/4 = -1/2 f(x) \qquad \approx \frac{\sqrt{3}}{4} - \frac{1}{2}(x - \pi/3) \\ f(\pi/2) &\approx \frac{\sqrt{3}}{4} - \frac{1}{2}(\pi/2 - \pi/3) = \frac{\sqrt{3}}{4} - \frac{\pi}{12}. \end{aligned}$$

Major Topic 4: Optimization

(a) We wish to build a rectangular pen with two parallel internal partitions, using 1000 feet of fencing. What dimensions maximize the total area of the pen?



Solution:

Our objective function is $A = \ell w$. We see also that $2\ell + 4w = 1000$ so we can write $\ell = 500 - 2w$, and thus we have

$$\begin{aligned} A &= (500 - 2w)w = 500w - 2w^2 \\ A' &= 500 - 4w \end{aligned}$$

has a critical point when $w = 125$.

We can see this is a maximum using the extreme value theorem: the function is defined on the interval $[0, 250]$, and $A(0) = A(250) = 0$.

Or we can use the first derivative test; we see that $A'(w) < 0$ when $w > 125$ and $A'(w) > 0$ when $w < 125$, so A has a local maximum at $w = 125$.

Or we can use the second derivative test. $A'' = -4 < 0$ so we have a local maximum.

Thus the pen is maximized with width 125 and length 250. (The maximum area, which I didn't ask for, is 31250 square feet.

(b) Classify all the critical points and relative extrema of $h(x) = x^3/(x + 1)$. (For each critical point, tell me whether it is a relative maximum, a relative minimum, or neither.)

Solution:

We have

$$h'(x) = \frac{3x^2(x+1) - x^3}{(x+1)^2} = \frac{3x^3 + 3x^2 - x^3}{(x+1)^2} = \frac{x^2(2x+3)}{(x+1)^2}$$

The critical points are thus at 0 and at $-3/2$, and a fake one at -1. We make a chart:

	x^2	$2x+3$	$(x+1)^{-2}$	$h'(x)$
$x < -3/2$	+	−	+	−
$-3/2 < x < -1$	+	+	+	+
$-1 < x < 0$	+	+	+	+
$0 < x$	+	+	+	+

This tells us that we have a local minimum at $x = -3/2$, and no other extrema. We compute $h(-3/2) = -27/8/(-1/2) = 27/4$, so the sole local minimum is $(-3/2, 27/4)$.

Alternatively we could use the second derivative test.

Secondary Topic 6: Curve Sketching

Sketch the graph of $f(x) = x^5 - 5x^4 + 5x^3 = x^3(x^2 - 5x + 5)$. We have $f'(x) = 5x^2(x-3)(x-1)$ and $f''(x) = 10x(2x^2 - 6x + 3)$.

You should discuss the domain, limits, critical points, intervals of increase and decrease, concavity, and possible points of inflection.

Solution: The domain of f is all reals.

There are roots at 0 and at $5/2 \pm \sqrt{5}/2$.

We see that $\lim_{x \to +\infty} f(x) = +\infty$ and $\lim_{x \to -\infty} f(x) = -\infty$.

The critical points are $0, 1, 3$. We compute $f(0) = 0, f(1) = 1, f(3) = -27$.

For increase and decrease we make a chart:

	$5x^2$	$(x-3)$	$(x-1)$	$f'(x)$
$x < 0$	+	−	−	+
$0 < x < 1$	+	−	−	+
$1 < x < 3$	+	−	+	−
$3 < x$	+	+	+	+

Thus f is increasing on $(-\infty, 1)$ and on $(3, +\infty)$, and is decreasing on $(1, 3)$.

The possible points of inflection are 0 and $\frac{6 \pm \sqrt{36-24}}{4} = \frac{3 \pm \sqrt{3}}{2}$. We can make a chart:

	$10x$	$2x^2 - 6x + 3$	$f'(x)$
$x < 0$	$-$	$+$	$-$
$0 < x < (3-\sqrt{3})/2$	$+$	$+$	$+$
$(3-\sqrt{3})/2 < x < (3+\sqrt{3}))/2$	$+$	$-$	$-$
$(3+\sqrt{3})/2 < x$	$+$	$+$	$+$

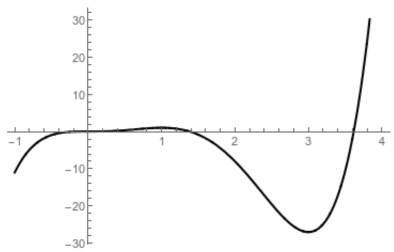


Figure 1: Graph of $f(x)$