Math 1231 Section 10 Fall 2021
Single-Variable Calculus I Mastery Quiz 9
Due Thursday, November 18

This week's mastery quiz has three topics. **You may submit all three**. This is the last week for topics M4 and S7.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

You shouldn't spend more than about 20-30 minutes on this quiz. Don't worry if you make a minor error, but try to demonstrate that you understand the concepts involved and have mastered the underlying material. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically through Blackboard but this should be a last resort.

Topics on This Quiz

- Major Topic 4: Extrema and Optimization
- Secondary Topic 7: Approximation
- Secondary Topic 8: Riemann Sums

Name:

Recitation Section:

M4: Extrema and Optimization

(a) Find the absolute extrema of $f(x) = 3x^4 - 20x^3 + 24x^2 + 7$ on $[0, 5]$.

Solution: f is a continuous function on a closed ineterval, so it must have an absolute maximum and an absolute minimum. $f'(x) = 12x^3 - 60x^2 + 48x = 12x(x^2 - 5x + 4) = 12x(x-4)(x-1)$ is defined everywhere and has roots at $0, 1, 4$. The endpoints are $0, 5$, so we need to evaluate f at $0, 1, 4, 5$.

$$f(0) = 7 \qquad f(1) = 14$$

$$f(4) = 3(4^4) - 5(4^4) + \frac{3}{2}(4^4) + 7 = \frac{-1}{2}4^4 + 7 = 7 - 128 = -121$$

$$f(5) = 3 \cdot 5^4 - 4 \cdot 5^4 + 5^4 - 5^2 + 7 = 7 - 25 = -18.$$

So the absolute maximum is 14 at 1, and the absolute minimum is -121 at 4.

(b) Classify all the critical points and relative extrema of $f(x) = \frac{x}{x^2+1}$. (For each critical point, tell me whether it is a relative maximum, a relative minimum, or neither.)

Solution:

We have

$$f'(x) = \frac{x^2 + 1 - 2x^2}{(x^2+1)^2} = \frac{1 - x^2}{(x^2+1)^2}$$

The critical points are thus at ± 1. We can make a chart:

	$(1-x)$	$1+x$	$(x^2+1)^{-2}$	$h'(x)$
$x < -1$	$+$	$-$	$+$	$-$
$-1 < x < 1$	$+$	$+$	$+$	$+$
$1 < x$	$-$	$+$	$+$	$-$

This tells us that we have a local minimum at $x = -1$ and a local maximum at $x = 1$.

Alternatively we could use the second derivative test. We have

$$f''(x) = \frac{-2x(x^2+1)^2 - 2(x^2+1)2x(1-x^2)}{(x^2+1)^4}$$

$$f''(-1) = \frac{8}{16} = 1/2 > 0$$

$$f''(1) = \frac{-8}{16} = -1/2 < 0$$

so we see that there's a local minimum at -1 and a local maximum at 1.

S7: Approximation

(a) Find a formula for the quadratic approximation of $f(x) = \frac{2x}{x+2}$ near the point $a = 2$, and use it to estimate $f(1.9)$.

Solution: We compute

$$f(1) = 1$$

$$f'(x) = \frac{2(x+2) - 2x}{(x+2)^2} = \frac{4}{(x+2)^2} \qquad f'(2) = 1/4$$

$$f''(x) = \frac{-8}{(x+2)^3} \qquad f''(2) = \frac{-1}{8}$$

and thus we have

$$\begin{aligned} f(x) &\approx 1 + \frac{1}{4}(x-2) - \frac{1}{16}(x-2)^2 \\ f(1.9) &\approx 1 - \frac{1}{4}(.1) - \frac{1}{16}(.1)^2 \\ &= \frac{1159}{1600} = 1 - .025 - .000625 = .974375 \end{aligned}$$

(b) Use two steps of Newton's method to estimate a solution to $\frac{2}{x} + 1 = x^2$ starting from $x_0 = 1$. (You should compute x_2.)

Solution: We're looking for a root of $g(x) = \frac{2}{x} - x^2 + 1$. So we compute $g'(x) = -\frac{2}{x^2} - 2x$, and we get

$$\begin{aligned} x_1 &= 1 - \frac{2}{-4} == \frac{3}{2} \\ x_2 &= \frac{3}{2} - \frac{g(3/2)}{g'(3/2)} = \frac{3}{2} - \frac{4/3 - 9/4 + 1}{-8/9 - 3} = \frac{3}{2} - \frac{1/12}{-35/9} = \frac{3}{2} + \frac{3}{140} = \frac{213}{140} \approx 1.5214285. \end{aligned}$$

(The true answer is approximately 1.5213797, so yay for us.)

Secondary Topic 8: Riemann Sums

Let $f(x) = x^2 - x$ be defined on the interval $[-3, 0]$.

(a) Approximate the area under the curve of the function using three rectangles and right endpoints.

(b) Approximate the area under the curve of the function using three rectangles and left endpoints.

(c) Find a formula for computing R_n, the estimate using n rectangles and right endpoints. (This formula should not have a summation sign or be given as a sum of n terms.)

(d) Use the formula in part (c) to compute the area exactly.

Solution:

(a) $R_3 = 1 \cdot f(-2) + 1 \cdot f(-1) + 1 \cdot f(0) = 6 + 2 + 0 = 8.$

(b) $L_3 = 1 \cdot f(-3) + 1 \cdot f(-2) + 1 \cdot f(-1) = 12 + 6 + 2 = 20.$

(c)

$$\begin{aligned}
R_n &= \sum_{i=1}^{n} \frac{3}{n} f\left(-3 + i\frac{3}{n}\right) = \sum_{i=1}^{n} \frac{3}{n} \left((3i/n - 3)^2 - (3i/n - 3)\right) \\
&= \sum_{i=1}^{n} \frac{3}{n} \left(\frac{9i^2}{n^2} - \frac{18i}{n} + 9 - \frac{3i}{n} + 3\right) \\
&= \sum_{i=1}^{n} \frac{27i^2}{n^3} - \frac{63i}{n^2} + \frac{36}{n} \\
&= \frac{27}{n^3} \sum_{i=1}^{n} i^2 - \frac{63}{n^2} \sum_{i=1}^{n} i + \frac{36}{n} \sum_{i=1}^{n} 1 \\
&= \frac{27}{n^3} \frac{n(n+1)(2n+1)}{6} - \frac{63}{n^2} \frac{n(n+1)}{2} + \frac{36}{n} n
\end{aligned}$$

(d) We can compute

$$\begin{aligned}
\lim_{n \to +\infty} R_n &= \lim_{n \to +\infty} \frac{27}{n^3} \frac{n(n+1)(2n+1)}{6} - \frac{63}{n^2} \frac{n(n+1)}{2} + \frac{36}{n} n \\
&= \lim_{n \to +\infty} \frac{27 \cdot 1(1 + 1/n)(2 + 1/n)}{6} - \frac{63 \cdot 1(1 + 1/n)}{2} + 36 \\
&= 9 - \frac{63}{2} + 36 = \frac{27}{2}.
\end{aligned}$$