# Math 1231 Section 16 Fall 2021
# Single-Variable Calculus I Mastery Quiz 9
# Due Monday, November 15

This week's mastery quiz has three topics. You may submit all three.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

You shouldn't spend more than about 20-30 minutes on this quiz. Don't worry if you make a minor error, but try to demonstrate that you understand the concepts involved and have mastered the underlying material. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in class on Monday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically through Blackboard but this should be a last resort.

**Topics on This Quiz**

- Major Topic 4: Optimization
- Secondary Topic 6: Curve Sketching
- Secondary Topic 7: Approximation

## Name:

## Recitation Section:

# Major Topic 4: Optimization

(a) Find the absolute extrema of $f(x) = x^3 + x^2 - 5x$ on $[-1, 2]$.

**Solution:** $f$ is a continuous function on a closed ineterval, so it must have an absolute maximum and an absolute minimum. $f'(x) = 3x^2 + 2x - 5 = (3x+5)(x-1)$ is defined everywhere and has roots at $-5/3$ and 1, so the critical points are $-5/3, 1$. We can ignore $-5/3$ because it isn't in the interval, so we need to evaluate $f$ at $-1, 1, 2$.

$$\begin{aligned} f(-1) &= 5 \\ f(1) &= -3 \\ f(2) &= 2 \end{aligned}$$

So the absolute minimum is $-3$ at 1 , and the absolute maximum is 5 at $-1$.

(b) Classify the critical points and relative extrema of $g(x) = \cos^2(x) - 2\sin(x)$ on $[0, 2\pi]$

**Solution:** We have

$$g'(x) = -2\cos(x)\sin(x) - 2\cos(x) = -2\cos(x)(\sin(x) + 1)$$

so $g'(x)$ is defined everywhere, and is 0 at $\pi/2, 3\pi/2$.

Here it looks easiest to use the second derivative test. We compute:

$$\begin{aligned} g''(x) &= 2\sin^2(x) - 2\cos^2(x) + 2\sin(x) \\ g''(\pi/2) &= 2 - 0 + 2 = 4 > 0 \\ g''(3pi/2) &= 2 - 0 - 2 = 0. \end{aligned}$$

So this tells us that $g$ has a local minimum at $\pi/2$, but doesn't tell us what happens at $3\pi/2$.

To answer that question we need the first derivative. If we make a chart we can plug in values like

$$\begin{aligned} g'(0) &= -2 \\ g'(\pi) &= 2 \\ g''(2\pi) &= -2 \end{aligned}$$

| | $g'(x)$ |
|---|---|
| $0 \le x < \pi/2$ | $-$ |
| $\pi/2 < x < 3\pi/2$ | $+$ |
| $3\pi/2 < x \le 2\pi$ | $-$ |

so $g$ has a relative minimum at $\pi/2$ and a relative maximum at $3\pi/2$.

# Secondary Topic 6: Curve Sketching

Let $g(x) = \dfrac{x^2-7}{x^2-4}$.

We can compute that $g'(x) = \dfrac{6x}{(x+2)^2(x-2)^2}$ and $g''(x) = \dfrac{-6(3x^2+4)}{(x^2-4)^3}$.

Sketch a graph of the function $g(x)$. Your answer should discuss the domain, asymptotes, roots, limits at infinity, critical points and values, intervals of increase and decrease, points of inflection, and concavity.

**Solution:**

$g$ has domain all reals except $\pm 2$. $g(x) = 0$ when $x = \pm\sqrt{7}$. $\lim_{x\to\pm\infty} \frac{x^2-7}{x^2-4} = \lim_{x\to\pm\infty} \frac{1-7/x^2}{1-4/x^2} = 1$.

The derivative $g'(x) = \frac{6x}{(x+2)^2(x-2)^2}$ is defined except at $x = \pm 2$. It is zero when $x = 0$. The critical points are thus 0 and arguably $\pm 2$; $g(0) = 7/4$.

Since the denominator is a square, it is positive when $x > 0$ and negative when $x < 0$, so it is decreasing for negative $x$ and increasing for positive $x$. This means that $(0, 7/4)$ is a relative minimum.

We know that

$$g''(x) = \frac{-6(3x^2+4)}{(x^2-4)^3}$$

The numerator is negative everywhere. The denominator is negative when $-2 < x < 2$ and is positive when $x < -2$ or when $2 < x$. Thus the concavity only changes at $\pm 2$; it is concave up on $(-2, 2)$ and concave down elsewhere.

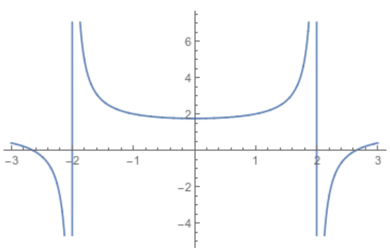


Figure 1: The graph of $g$ from $-3$ to $3$

# Secondary Topic 7: Approximation

(a) Find a formula for the quadratic approximation of $f(x) = \sin(x^2 + x)$ near the point $a = 0$, and use it to estimate $f(.1)$.

**Solution:** We compute

$$\begin{aligned} & & f(0) &= 0 \\ f'(x) &= \cos(x^2+x)(2x+1) & f'(0) &= 1 \\ f''(x) &= -\sin(x^2+x)(2x+1)^2 + 2\cos(x^2+x) & f''(0) &= 2 \end{aligned}$$

and thus we have

$$f(x) \approx 0 + 1(x) + \frac{2}{2}(x)^2 = x + x^2$$
$$f(.1) \approx .1 + .01 = .11.$$

(b) Use two steps of Newton's method to estimate a solution to $x^3 + x = 1$ starting from $x_0 = 1$. (You should compute $x_2$.)

**Solution:** We're looking for a root of $g(x) = x^3 + x - 1$. So we compute $g'(x) = 3x^2 + 1$, and we get

$$x_1 = 1 - \frac{g(1)}{g'(1)} = 1 - \frac{1}{4} = \frac{3}{4}$$
$$x_2 = \frac{3}{4} - \frac{g(3/4)}{g'(3/4)} = \frac{3}{4} - \frac{27/64 + 3/4 - 1}{27/16 + 1} = \frac{3}{4} - \frac{11/64}{43/16} = \frac{3}{4} - \frac{11}{172} = \frac{118}{172} \approx .686.$$

(The true answer is approximately .68233, so yay for us.)