# Math 2233 Midterm 1 Solutions

Instructor: Jay Daigle

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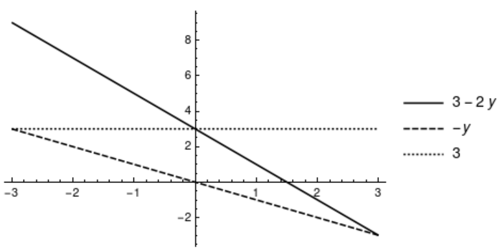
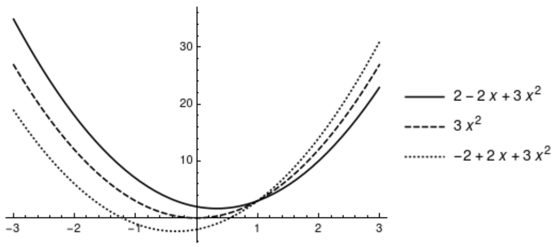
**Problem 1.** Let $f(x, y) = 3x^2 + xy - y$

(a) Sketch cross-sections of $f$ for $x = -1, 0, 1$ and $y = -2, 0, 2$.




**Solution:**




$$f(-1, y) = 3 - y - y = 3 - 2y \qquad f(0, y) = -y \qquad f(1, y) = 3 + y - y = 3$$
$$f(x, -2) = 3x^2 - 2x + 2 \qquad f(x, 0) = 3x^2 \qquad f(x, 2) = 3x^2 + 2x - 2$$

(b) If $\vec{u} = \frac{2}{3}\vec{i} + \frac{\sqrt{5}}{3}\vec{j}$, compute $f_{\vec{u}}(0, 2)$.

**Solution:** $\nabla f(x, y) = (6x + y)\vec{i} + (x - 1)\vec{j}$, so

$$\nabla f(0, 2) = 2\vec{i} - \vec{j}$$
$$f_{\vec{u}}(0, 2) = 2 \cdot \frac{2}{3} - \frac{\sqrt{5}}{3} = \frac{4 - \sqrt{5}}{3}.$$

(c) At the point $(1, 3)$, in what direction should you move to increase $f(x, y)$ at the fastest possible rate? What is the rate of increase in that direction?

**Solution:** The direction of greatest increase is $\nabla f(1, 3) = 9\vec{i}$. The rate of increase is $\|\nabla f(1, 3)\| = 9$.

**Problem 2.** (a) Give an equation for a plane through the points $(0, 0, 1), (2, 3, 2), (4, 1, -2)$.

**Solution:** There are two approaches here.

First, we know that we have

$$z = 1 + m(x - 0) + n(y - 0) = 1 + mx + ny.$$

Then we get

$$\begin{aligned} 2 &= 1 + 2x + 3y \\ -2 &= 1 + 4x + y \\ -6 &= -1 - 5y \end{aligned}$$

This tells us $y = 1$, and then we conclude $x = -1$, so we get

$$z = 1 - x + y.$$

Alternatively, we get the vectors $2\vec{i} + 3\vec{j} + \vec{k}$ and $4\vec{i} + \vec{j} - 2\vec{k}$. Then we compute

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 3 & 1 \\ 4 & 1 & -3 \end{vmatrix} = -9\vec{i} + 4\vec{j} + 2\vec{k} - 12\vec{k} - \vec{i} + 6\vec{j} = -10\vec{i} + 10\vec{j} - 10\vec{k}$$

and thus the equation for the plane is

$$0 = -10(x - 0) + 10(y - 0) - 10(z - 1)$$

or

$$-10x + 10y - 10z + 10 = 0.$$

These are, non-obviously, the same plane.

(b) Find the cosine of the angle between the vectors $\vec{v} = 6\vec{i} - 2\vec{j} + 2\vec{k}$ and $\vec{u} = 4\vec{i} - 1\vec{j} + 2\vec{k}$.

**Solution:** We know that

$$\cos\theta = \frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\| \cdot \|\vec{u}\|} = \frac{-6 - 1 - 2}{\sqrt{11}\sqrt{9}} = \frac{-9}{3\sqrt{11}} = \frac{-3}{\sqrt{11}}.$$

(c) Let $\vec{v} = 2\vec{i} + 3\vec{j} + 2\vec{k}$ and $\vec{u} = \vec{i} - \vec{j} + \vec{k}$. Compute the orthogonal decomposition of $\vec{v}$ with respect to $\vec{u}$. That is, write $\vec{v} = \vec{v}_{\text{parallel}} + \vec{v}_{\perp}$.

**Solution:**

$$\begin{aligned} \vec{v}_{\text{parallel}} &= \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u} = \frac{2 - 3 + 2}{1 + 1 + 1}\vec{u} \\ &= \frac{1}{3}\vec{u} = \frac{1}{3}\vec{i} - \frac{1}{3}\vec{j} + \frac{1}{3}\vec{k} \\ \vec{v}_{\perp} &= \vec{v} - \vec{v}_{\text{parallel}} = \frac{5}{3}\vec{i} + \frac{10}{3}\vec{j} + \frac{5}{3}\vec{k}. \end{aligned}$$

**Problem 3.** (a) Compute $\nabla \ln(ye^{xy})$.

**Solution:**

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{y^2 e^{xy}}{ye^{xy}} = y \\ \frac{\partial z}{\partial y} &= \frac{e^{xy} + ye^{xy}x}{ye^{xy}} = \frac{1 + xy}{y} \\ \nabla z &= y\vec{i} + \frac{1 + xy}{y}\vec{j} \end{aligned}$$

(b) Find an equation for the tangent plane to the graph of the function $f(x,y) = 2 + \sin(xy) + \frac{y}{x}$ at the point $(1,0)$.

**Solution:** We have

$$\nabla f(x,y) = \left(y\cos(xy) - \frac{y}{x^2}\right)\vec{i} + \left(x\cos(xy) + \frac{1}{x}\right)\vec{j},$$

so $\nabla f(1,0) = 2\vec{j}$.

Further we have $f(2,0) = 2$. Thus we get the equation

$$z = 2 + 2(y - 0).$$

(c) Consider the function $g(x) = e^x \cos(y)$. Both the values $g(0.1, 0) = e^{0.1}$ and $g(0, 0.1) = \cos(0.1)$ should be close in value to $g(0,0) = 1$. Which of these values do you think is closer to $g(0,0)$, and why?

(Your argument should *not* involve the use of a calculator.)

**Solution:**

$$\begin{aligned}
\nabla g(x,y) &= e^x \cos(y)\vec{i} - e^x \sin(y)\vec{j} \\
\nabla g(0,0) &= \vec{i} - 0\vec{j} \\
g(x,y) &\approx 1 + 1(x-0) = 1 + x \\
g(0.1,0) &\approx 1.1 \\
g(0,0.1) &\approx 1
\end{aligned}$$

So we expect $g(0,0.1)$ to be closer to $g(0,0)$ than $g(0.1,0)$ is.

**Problem 4.** Let $f(x,y) = \sin(x^2 y) + xy^2$.

(a) Find the degree 2 Taylor polynomial for $f$ centered at $(2,0)$.

**Solution:**

We compute

$$\begin{aligned}
f_x(x,y) &= 2xy\cos(x^2y) + y^2 & f_x(2,0) &= 0 \\
f_y(x,y) &= x^2\cos(x^2y) + 2xy & f_y(2,0) &= 4 \\
f_{xx}(x,y) &= 2y\cos(x^2y) - 4x^2y^2\sin(x^2y) & f_{xx}(2,0) &= 0 \\
f_{xy}(x,y) &= 2x\cos(x^2y) - 2x^3y\sin(x^2y) + 2y & f_{xy}(2,0) &= 4 \\
f_{yy}(x,y) &= -x^4\sin(x^2y) + 2x & f_{yy}(2,0) &= 4.
\end{aligned}$$

Thus the Taylor polynomial is

$$\begin{aligned}
T_2(x,y) &= 0 + 0(x-2) + 4(y-0) + 0(x-2)^2 + 4(x-2)(y-0) + 2(y-0)^2 \\
&= 4y + 4y(x-2) + 2y^2.
\end{aligned}$$

(b) Use your answer in part (a) to estimate $f(2.2, -.1)$.

**Solution:** We have

$$\begin{aligned}
f(2.2,-.1) &\approx 4(-.1) + 4(-.1)(.2) + 2(-.1)^2 \\
&= -.4 - .08 + .02 = -.46.
\end{aligned}$$

**Problem 5.** (a) Find and classify the critical points of $f(x,y) = 8x - 8y + 2xy - x^2 + 2y^3$.

**Solution:** We have

$$f_x(x, y) = 8 + 2y - 2x$$
$$f_y(x, y) = -8 + 2x + 6y^2.$$

The second equation tells us that $x = 4 - 3y^2$, and plugged into the first equation that gives

$$0 = 8 + 2y - 8 + 6y^2 = 2y + 6y^2$$

so either $y = 0$ or $y = -1/3$. If $y = 0$ then we have $x = 4$; if $y = 1/3$ then $x = 4 - 1/3 = 11/3$. Thus our critical points are $(4, 0)$ and $(11/3, -1/3)$.

We have

$$f_{xx}(x, y) = -2 \qquad f_{xx}(4, 0) = -2 \qquad f_{xx}(11/3, -1/3) = -2$$
$$f_{xy}(x, y) = 2 \qquad f_{xy}(4, 0) = 2 \qquad f_{xy}(11/3, -1/3) = 2$$
$$f_{yy}(x, y) = 12y \qquad f_{yy}(4, 0) = 0 \qquad f_{yy}(11/3, -1/3) = -4.$$

Then for $(4, 0)$ we have $D = (-2) \cdot 0 - 2^2 = -4 < 0$, so we have a saddle point.

For $(11/3, -1/3)$ we have $D = -2 \cdot (-4) - 2^2 = 4 > 0$, and $f_{xx}(4, 0) = -2 < 0$. So this is a local maximum.

(b) Find (but don't classify) the critical points of $g(x, y, z) = y + z + x^2z - xyz$.

**Solution:** We have

$$g_x(x, y, z) = 2xz - yz$$
$$g_y(x, y, z) = 1 - xz$$
$$g_z(x, y, z) = 1 + x^2 - xy$$

The second equation says that $z = 1/x$. Plugging that into the first equation gives $2 - y/x = 0$ and thus $y = 2x$. Then the third equation gives us $0 = 1 + x^2 - 2x^2$ and thus $x^2 = 1$ and so $x = \pm 1$.

If $x = 1$ then $y = 2$ and $z = 1$. If $x = -1$ then $y = -2$ and $z = -1$. So the two critical points are $(1, 2, 1)$ and $(-1, -2, -1)$.