Math 2233 Midterm 2 Solutions

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Problem 1. (a) Use Lagrange Multipliers to find the minimum value of $f(x, y) = xy$ subject to the constraint $4x^2 + y^2 = 8$.

Solution:

We have

$$y = 8\lambda x$$
$$x = 2\lambda y$$

This implies that $y = 16\lambda^2 y$ and thus $\lambda = \pm 1/4$. Then $y = \pm 2x$ and our constraint gives us

$$\begin{aligned} 4x^2 + (\pm 2x)^2 &= 8 \\ 8x^2 &= 8 \\ x &= \pm 1 \end{aligned}$$

so our four critical points are $(1, 2), (1, -2), (-1, 2), (-1, -2)$. (And we can check that all four of these do in fact work.)

Now we compute

$$\begin{aligned} f(1, 2) &= 2 \\ f(1, -2) &= -2 \\ f(-1, 2) &= -2 \\ f(-1, -2) &= 2. \end{aligned}$$

So the absolute maximum is 2 and the absolute minimum is -2.

(b) Find a parametrization for a cylinder, centered on the line $x = 1, z = 2$, with radius 4.

Solution:

For each y we have a circle of radius 4 centered at the point $(1, y, 2)$, which we can parameterize as $(4\cos(t) + 1, y, 4\sin(t) + 2)$. Thus our total parameterization can be

$$\vec{r}(s, t) = (4\cos(t) + 1, s, 4\sin(t) + 2).$$

Problem 2. (a) Sketch the region of integration and compute the integral $\iint_R \sin(\pi x^2)\,dA$ over the region bounded by $x + y = 0, x - 2y = 0$, and $x = 3$.

Solution:

We have

$$\begin{aligned} I &= \int_0^3 \int_{-x}^{x/2} \sin(\pi x^2)\,dy\,dx \\ &= \int_0^3 y\sin(\pi x^2)\Big|_{-x}^{x/2}\,dx \\ &= \int_0^3 \frac{3x}{2}\sin(\pi x^2)\,dx \\ &= -3\cos(\pi x^2)/4\pi\Big|_0^3 = \frac{-3}{4\pi}(\cos(9\pi) - \cos(0)) = \frac{6}{4\pi} = \frac{3}{2\pi}. \end{aligned}$$

(b) Use a change of variables to evalutate $\iint_T \sin\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)\,dA$ where T is the triangle with vertices $(0,0), (4,0)$, and $(2,2)$.

Solution: It seems reasonable to take $s = (x+y)/2$ and $t = (x-y)/2$, so the integrand becomes $\sin(s)\sin(t)$. Then we have $x = s+t$ and $y = s-t$, and our triangle has vertices $(0,0), (2,2), (2,0)$. The Jacobian is

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} 1 & 1 \\ 1 & -1 \end{vmatrix} = -1-1 = -2.$$

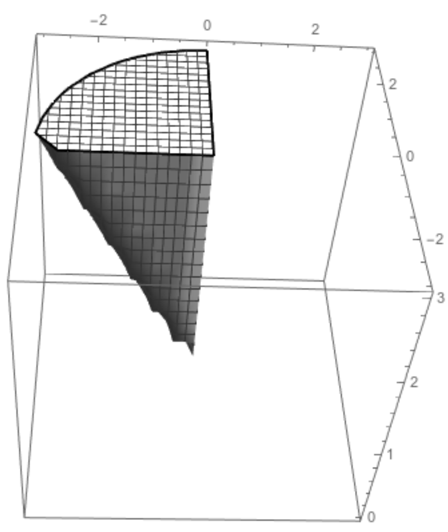
Thus our integral is

$$\begin{aligned}
\int_0^2 \int_0^s \sin(s)\sin(t)\cdot|-2|\,dt\,ds &= 2\int_0^2 -\sin(s)\cos(t)\Big|_0^s\,ds \\
&= -2\int_0^2 \sin(s)(\cos(s)-1)\,ds \\
&= 2\int_0^2 \sin(s)-\sin(s)\cos(s)\,ds \\
&= 2\left(-\cos(s)+\cos^2(s)/2\right)\Big|_0^2 \\
&= 2\left(-\cos(2)+\cos^2(2)/2+\cos(0)-\cos^2(0)/2\right) \\
&= 1-2\cos(2)+\cos^2(2).
\end{aligned}$$

Problem 3. Let R be the quarter-cone in the second quadrant $x \le 0, y \ge 0$ bounded by $z = 2$, and the cone $z = \sqrt{x^2+y^2}$ (as shown below). Let $f(x,y,z) = xy$.

(a) Set up integrals to compute $\int_R f\,dA$ in cartesian, cylindrical, and spherical coordinates.

(b) Choose one of these integrals and evaluate it.



Solution:

(a) For Cartesian coordinates: the top of the cone is the circle $x^2+y^2=4$, but we only have the second quadrant. So we can take $-2 \le x \le 0$ and then $0 \le y \le \sqrt{4-x^2}$, and we have $\sqrt{x^2+y^2} \le z \le 2$. This gives

$$\int_{-2}^0 \int_0^{\sqrt{4-x^2}} \int_{\sqrt{x^2+y^2}}^2 xy\,dz\,dy\,dx.$$

For cylindrical coordinates: We have a radius from 0 to 3, and our angle goes from $\pi/2$ to π. Then we have $r \leq z \leq 2$. We know that $x = r\cos\theta$ and $y = r\sin\theta$ so we have the integral

$$\int_0^2 \int_{\pi/2}^{\pi} \int_r^2 r^2 \sin\theta \cos\theta \cdot r\, dz\, d\theta\, dr.$$

For spherical coordinates: the side of the cone is at a 45-degree angle so we have $0 \leq \phi \leq \pi/4$. We still have $\pi/2 \leq \theta \leq \pi$. To calculate ρ we look at the cone from the side to see a right triangle; we get $\cos\phi = 2/\rho$ and thus $\rho = 2\sec\phi$. Since $x = \rho\sin(\phi)\cos(\theta)$ and $y = \rho\sin(\phi)\sin(\theta)$ we get the integral

$$\int_0^{\pi/4} \int_{\pi/2}^{\pi} \int_0^{2\sec(\phi)} \rho^2 \sin^2(\phi)\cos(\theta)\sin(\theta) \cdot \rho^2 \sin(\phi)\, d\rho\, d\theta\, d\phi.$$

(b) I really hope everyone picks the cylindrical integral. We compute

$$\begin{aligned}
I &= \int_0^2 \int_{\pi/2}^{\pi} \int_r^2 r^3 \sin\theta\cos\theta \cdot\, dz\, d\theta\, dr \\
&= \int_0^2 \int_{\pi/2}^{\pi} 2r^3 \sin(\theta)\cos(\theta) - r^4 \sin(\theta)\cos(\theta)\, d\theta\, dr \\
&= \int_0^2 r^3 \sin^2(\theta) - r^4 \sin^2(\theta)/2 \Big|_{\pi/2}^{\pi} dr \\
&= \int_0^2 r^4/2 - r^3\, dr \\
&= r^5/10 - r^4/4 \Big|_0^2 = 16/5 - 4 = -4/5.
\end{aligned}$$

Problem 4. (a) Find a parametric equation for a particle moving in a straight line from $(1, 7, -4)$ to $(4, 4, 2)$

Solution:

$$\vec{r}(t) = (1, 7, -4) + t(3, -3, 6) = (1 + 3t, 7 - 3t, -4 + 6t).$$

(b) Suppose another particle follows the path $\vec{r}_2(t) = (4t, t + 3, t^2 + t)$. Does this particle's path intersect the path of the particle from part (a)?

Solution:

We would need

$$\begin{aligned}
1 + 3t_1 &= 4t_2 \\
7 - 3t_1 &= t_2 + 3 \\
6t_1 - 4 &= t_2^2 + t_2
\end{aligned}$$

The second equation gives that $t_2 = 4 - 3t_1$. Plugging that into the first equation gives $1 + 3t_1 = 16 - 12t_1$ and thus $t_1 = 1$, so $t_2 = 1$ as well. Then we see that $t_1 = t_2 = 1$ satisfies the third equation as well, so the particles' paths do intersect—and in fact the particles themselves collide, since it happens at the same time.

(c) Find a flow line for the vector field $\vec{F}(x, y) = 2x\vec{i} + x\vec{j}$ that goes through the point $(2, 4)$.

Solution: If $\vec{r}'(t) = \vec{F}(\vec{r}(t))$, we have $x'(t) = 2x(t)$ and $y'(t) = x(t)$, which means $x(t) = C_1 e^{2t}$ and then $y(t) = C_1 e^{2t}/2 + C_2$.

If we assume this path goes through $(1, 1)$ at time $t = 0$ then we have $2 = C_1 \cdot e^0 = C_1$ and $4 = 2e^0/2 + C_2$ giving us $C_2 = 3$. Thus $\vec{r}(t) = (2e^{2t}, e^{2t} + 3)$.