# Math 2233 Practice Final Solutions

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- These are the instructions you will see on the real test, next week. I include them here so you know what to expect.
- This practice test is too long. The real test will be similar but have fewer questions; this is nine pages and I want to write a real final of about six pages. But I wanted to give you more practice, rather than less.
- You will have 120 minutes for the real final.
- You are not allowed to consult books or notes during the test, but you may use a one-page, two-sided, handwritten cheat sheet you have made for yourself ahead of time.
- You may use a calculator, but don't use a graphing calculator or anything else that can do symbolic computations. Using a calculator for basic arithmetic is fine.

**Problem 1.** (15 points each)

(a) Find a linear approximation of $f(x, y) = \sin(x)\sqrt{1 - y^2}$ near the point $(0, 0)$. Use it to estimate $f(.1, .1)$.

**Solution:**

$$\begin{aligned}
\nabla f(x, y) &= (\cos(x)\sqrt{1 - y^2}, \sin(x)y/\sqrt{1 - y^2} \\
\nabla f(0, 0) &= (1, 0) \\
f(x, y) &\approx 0 + 1(x - 0) + 0(y - 0) = x \\
f(.1, .1) &\approx .1.
\end{aligned}$$

(b) Find and classify all the critical points of $g(x, y) = x^2 - 3xy + 5x - 2y + 6y^2 + 8$.

**Solution:**

$$\begin{aligned}
g_x(x, y) &= 2x - 3y + 5 \\
g_y(x, y) &= -3x + 12y - 2 \\
0 &= -9y + 15 + 24y - 4 = 15y + 11
\end{aligned}$$

so we see that $y = -11/15$ and $x = -18/5$. This is the only critical point. The second derivatives are

$$\begin{aligned}
g_{xx}(x, y) &= 2 > 0 \\
g_{xy}(x, y) &= -3 \\
g_{yy}(x, y) &= 12 \\
D &= g_{xx}g_{yy} - g_{xy}^2 = 24 - 9 = 15 > 0
\end{aligned}$$

so this point is a local minimum.

(c) Find the minimum value of $f(x, y) = 4xy$ on the unit circle.

**Solution:** Our constraint equation is $x^2 + y^2 = 1$. So we have:

$$\begin{aligned} 4y &= \lambda 2x \\ 4x &= \lambda 2y \\ \lambda &= 2y/x \\ 4x &= 4y^2/x \\ 4x^2 &= 4y^2 \\ x^2 &= y^2 \\ x &= \pm y \end{aligned}$$

Plugging either of these into our constraint equation gives $2x^2 = 1$ and thus $x = \pm\sqrt{1/2}$. Thus we have four critical points: $(\sqrt{1/2}, \sqrt{1/2}), (\sqrt{1/2}, -\sqrt{1/2}), (-\sqrt{1/2}, \sqrt{1/2}), (-\sqrt{1/2}, -\sqrt{1/2})$. Plugging these in gives $2, -2, -2, 2$ respectively. So the absolute minimum value is $-2$.

**Problem 2.** (15 points each) Let

$$\vec{F}(x, y, z) = (0, x, y) \qquad \vec{G}(x, y, z) = (2x, z, y) \qquad \vec{H}(x, y, z) = (3y, 2x, z).$$

(a) For each field, either find a scalar potential function or prove that none exists.

**Solution:** We have

$$\nabla \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & x & y \end{vmatrix} = (1-0)\vec{i} + (0-0)\vec{j} + (1-0)\vec{k} \neq \vec{0}$$

$$\nabla \times \vec{G} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x & z & y \end{vmatrix} = (1-1)\vec{i} + (0-0)\vec{j} + (0-0)\vec{k} = \vec{0}$$

$$\nabla \times \vec{H} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3y & 2x & z \end{vmatrix} = (0-0)\vec{i} + (0-0)\vec{j} + (2-3)\vec{k} \neq \vec{0}$$

so the only field that could be conservative is $\vec{G}$. To find a potential function, we would need

$$\begin{aligned} \frac{\partial g}{\partial x} &= 2x \\ \frac{\partial g}{\partial y} &= z \\ \frac{\partial g}{\partial z} &= y \end{aligned}$$

The first equation tells us $g(x, y, z) = x^2 + h(y, z)$. The second tells us that $g(x, y, z) = yz + i(x, z)$ and the third tells us that $g(x, y, z) = yz + j(x, y)$. Putting this all together, we can take $g(x, y, z) = x^2 + yz$.

(b) For each field, either find a vector potential function or prove that none exists.

**Solution:** $\nabla \cdot \vec{F} = 0$ so $\vec{F}$ is irrotational. We set up a system

$$\begin{aligned} -\frac{\partial F_2}{\partial z} &= 0 \\ \frac{\partial F_1}{\partial z} &= x \\ \frac{\partial F_2}{\partial x} - \frac{\partial F_1}{\partial y} &= y. \end{aligned}$$

The first equation tells us that $F_2 = g(x,y)$, and the second equation tells us that $F_1 = xz + h(x,y)$. Then the third equation tells us that $g_x(x,y) - h_y(x,y) = y$; one reasonable solution for this is $g(x,y) = xy$. Thus $\vec{F}$ has a vector potential of $(xz, xy, 0)$.

$\nabla \cdot \vec{G} = 2$, so $\vec{G}$ is not a curl field. $\nabla \cdot \vec{H} = 1$, so $\vec{H}$ is not a curl field.

(c) Let $\vec{r}(t) = (2, 2t, t^2)$. For which of these vector fields is $\vec{r}$ a flow line? Justify your answer.

**Solution:**

$$\begin{aligned}
\vec{r}\,'(t) &= (0, 2, 2t) \\
\vec{F}(\vec{r}(t)) &= (0, 2, 2t) = \vec{r}\,'(t) \\
\vec{G}(\vec{r}(t)) &= (4, t^2, 2t) \neq \vec{r}\,'(t) \\
\vec{H}(\vec{r}(t)) &= (6t, 4, t^2) \neq \vec{r}\,'(t).
\end{aligned}$$

Thus $\vec{r}$ is a flow line of $\vec{F}$, and not of $\vec{G}$ or $\vec{H}$.

**Problem 3.** (15 points each) Let $g(x,y,z) = z(x^2 + y^2)$ and let $W$ be a cone with its point at the origin and base given by the circle $z = 2, x^2 + y^2 = 2$.

(a) Set up integrals to compute $\int_W g\, dV$ in cartesian, cylindrical, and spherical coordinates.

**Solution:**

$$\int_0^2 \int_{-z/\sqrt{2}}^{z/\sqrt{2}} \int_{-\sqrt{z^2/2 - x^2}}^{\sqrt{z^2/2-x^2}} z(x^2+y^2)\, dy\, dx\, dz \quad \text{or} \quad \int_{-\sqrt{2}}^{\sqrt{2}} \int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}} \int_{\sqrt{2(x^2+y^2)}}^{2} z(x^2+y^2)\, dz\, dy\, dx$$

$$\int_0^2 \int_0^{z/\sqrt{2}} \int_0^{2\pi} zr^2 \cdot r\, d\theta\, dr\, dz \quad \text{or} \quad \int_0^{\sqrt{2}} \int_{\sqrt{2}r}^{2} \int_0^{2\pi} zr^2 \cdot r\, d\theta\, dz\, dr$$

$$\begin{aligned}
&\int_0^{\arctan(1/\sqrt{2})} \int_0^{2\pi} \int_0^{2/\cos\phi} \rho\cos\phi(\rho^2 \sin^2\phi \cos^2\theta + \rho^2 \sin^2\phi \sin^2\theta \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi \\
&= \int_0^{\arctan(1/\sqrt{2})} \int_0^{2\pi} \int_0^{2/\cos\phi} \rho\cos\phi(\rho^2\sin^2\phi)\rho^2 \sin\phi\, d\rho\, d\theta\, d\phi
\end{aligned}$$

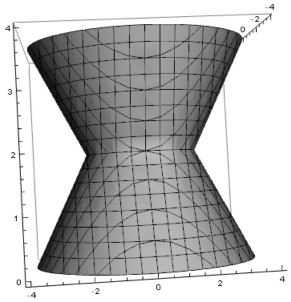
(b) Choose one of the integrals from part (a) and evaluate it.

**Solution:** The cylindrical coordinates are probably the easiest to work with. We compute

$$\begin{aligned}
\int_0^2 \int_0^{z/\sqrt{2}} \int_0^{2\pi} r^3 z\, d\theta\, dr\, dz &= 2\pi \int_0^2 \int_0^{z/\sqrt{2}} r^3 z\, dr\, dz \\
&= 2\pi \int_0^2 \frac{r^4}{4} z|_0^{z/\sqrt{2}}\, dz = 2\pi \int_0^2 z^5/16\, dz \\
&= 2\pi z^6/96|_0^2 = 128\pi/96 = 4\pi/3.
\end{aligned}$$

**Problem 4.** (10 points each) Set up but **do not evaluate** an integral to answer each of the following questions. Each answer should be an iterated integral containing no vector operations and no variables other than the variables of integration.

(a) Find the volume of the following shape made up of two cones squashed together, which has its base at $z = 0$, its top at $z = 4$, and has a radius of 4 at the base and top, and a radius of 2 at the thinnest point at $z = 2$.

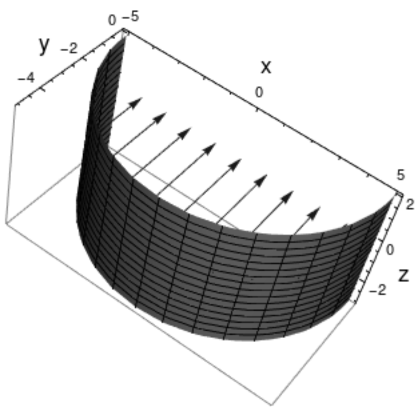


**Solution:**

$$\int_0^2 \int_0^{2\pi} \int_0^{4-z} r\,dr\,d\theta\,dz + \int_2^4 \int_0^{2\pi} \int_0^{z} r\,dr\,d\theta\,dz$$

$$\int_0^2 \int_{-\sqrt{16-(4-z)^2}}^{\sqrt{16-(4-z)^2}} \int_{-\sqrt{16-(4-z)^2-x^2}}^{\sqrt{16-(4-z)^2-x^2}} dy\,dx\,dz + \int_2^4 \int_{-\sqrt{16-z^2}}^{\sqrt{16-z^2}} \int_{-\sqrt{16-z^2-x^2}}^{\sqrt{16-z^2-x^2}} dy\,dx\,dz$$

(b) What is the flux of the vector field $\vec{F}(x,y,z) = xy\vec{i} + xz\vec{j} + yz\vec{k}$ through the $y \leq 0$ half of the side of a cylinder of radius 5, centered at the $z$ axis, which goes from $z = -3$ to $z = 2$, oriented towards the $z$-axis?




**Solution:**

$$\int_{-3}^{2} \int_{\pi}^{2\pi} (25\sin\theta\cos\theta, 5z\cos\theta, 5z\sin\theta) \cdot -(\cos\theta, \sin\theta, 0)5\,dz\,d\theta$$

$$= \int_{-3}^{2} \int_{\pi}^{2\pi} -125\sin\theta\cos^2\theta - 25z\sin\theta\cos\theta\,d\theta\,dz$$

(c) What is the work done by the force field $\vec{G}(x,y,z) = \sin(xz)y\vec{i} + e^{xyz}\vec{j} + \sqrt{x+y+z}\vec{k}$ on a particle following the path $\vec{r}(t) = (t, t^2, t^4)$ from time $t = 0$ to time $t = 5$.

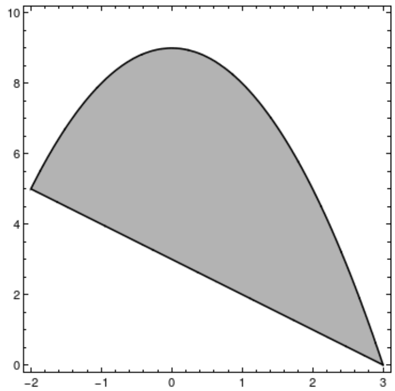
**Solution:**

$$\int_0^5 \vec{G}(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_0^5 (\sin(t^5)t^2, e^{t^7}, \sqrt{t+t^2+t^4})\cdot(1, 2t, 4t^3)\,dt = \int_0^5 \sin(t^5)t^2 + 2te^{t^7} + 4t^3\sqrt{t+t^2+t^4}\,dt.$$

(d) Integrate the function $f(x,y) = 5xy^3$ over the region bounded by $y = 9 - x^2$ and $y = 3 - x$. Sketch the region of integration.

**Solution:** $9 - x^2 = 3 - x$ when $x^2 - x - 6 = 0$ so when $(x-3)(x+2) = 0$.

$$\int_{-2}^{3}\int_{3-x}^{9-x^2} 5xy^3\,dy\,dx.$$

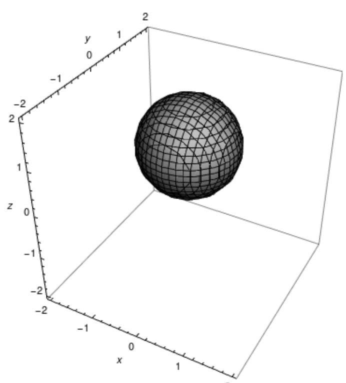



(e) What is the surface area of the graph of $f(x,y) = e^{xy} + \sin(x)\cos(y)$ for $0 \le x \le 3$ and $1 \le y \le \pi$?

**Solution:**

$$\int_0^3 \int_1^\pi \sqrt{1 + (ye^{xy} + \cos(x)\cos(y))^2 + (xe^{xy} - \sin(x)\sin(y))^2}\,dy\,dx$$

(f) Find the mass of a solid spherical ball of radius 1 centered at the point $(0,0,1)$ if its density is given by $\delta(x,y,z) = x^2 z$.

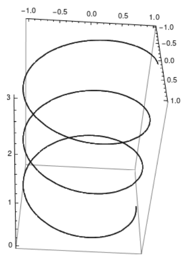



**Solution:**

$$\int_0^2 \int_{-\sqrt{1-(z-1)^2}}^{\sqrt{1-(z-1)^2}} \int_{-\sqrt{1-x^2-(z-1)^2}}^{\sqrt{1-x^2-(z-1)^2}} x^2 z\,dy\,dx\,dz$$

$$\int_0^1 \int_0^{2\pi} \int_0^\pi (\rho\cos\theta\sin\phi)^2(\rho\cos\phi + 1)\rho^2 \sin\phi\,d\phi\,d\theta\,d\rho$$

(g) Set up an integral to compute the work done by the force field $\vec{F}(x^2y, yz^3, x+y+z)$ on a particle that moves from $(1,0,0)$ to $(1,0,3)$ by spiraling clockwise around the $z$-axis three times with radius 1.

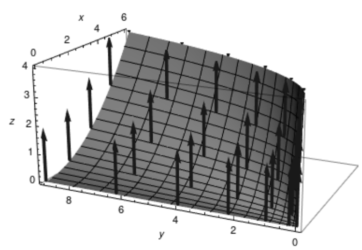


**Solution:** We can parametrize with $\vec{r}(t) = (\cos(2\pi t), -\sin(2\pi t), t)$ for $t \in [0,3]$. (This makes sure we both move clockwise and start at $(1,0,0)$; the $2\pi$ is to make a change of 1 in $t$ cause a complete rotation.) Then the integral is

$$\int_0^3 \big(-\cos^2(2\pi t)\sin(2\pi t), -\sin(2\pi t)t^3, (\cos(2\pi t) - \sin(2\pi t) + t)\big) \cdot (-2\pi\sin(2\pi t), -2\pi\cos(2\pi t), 1)\,dt$$
$$= \int_0^3 2\pi\sin^2(2\pi t)\cos^2(2\pi t) + 2\pi\sin(2\pi t)\cos(2\pi t)t^3 + \cos(2\pi t) - \sin(2\pi t) + t\,dt.$$

(h) Find the flux of the vector field $\vec{F}(x,y,z) = (x, xy, z)$ through the surface parametrized by $\vec{r}(s,t) = (st, s^2, t^2)$ oriented upwards, for $0 \le s \le 3, 0 \le t \le 2$.

*Note: the arrows in the diagram are the orientation of the surface, not a representation of $F$.*




**Solution:** We need the normal vector. We have

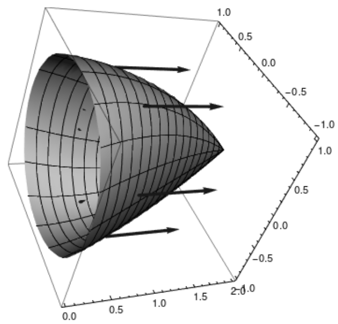
$$\vec{r}_s = (t, 2s, 0)$$
$$\vec{r}_t = (s, 0, 2t)$$
$$\vec{r}_s \times \vec{r}_t = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ t & 2s & 0 \\ s & 0 & 2t \end{vmatrix} = (4st - 0)\vec{i} + (0 - 2t^2)\vec{j} + (0 - 2s^2)\vec{k}$$

is oriented downwards, so instead we take $-4st\vec{i} + 2t^2\vec{j} + 2s^2\vec{k}$. Then the integral is

$$\int_0^3 \int_0^2 (st, s^3t, t^2) \cdot (-4st, 2t^2, 2s^2)\,dt\,ds$$
$$= \int_0^3 \int_0^2 -4s^2t^2 + 2s^3t^3 + 2s^2t^2\,dt\,ds$$
$$= \int_0^3 \int_0^2 2s^3t^3 - 2s^2t^2\,dt\,ds.$$

**Problem 5.** (20 points each) Compute (and evaluate!) each of the following integrals. You may often wish to use a theorem or other result to replace the given integral with an easier integral. Please identify the result you are using.

(a) Let $\vec{F}(x,y,z) = \sqrt{x^5 + x}\vec{i} + (x^2yz - z)\vec{j} + (x\sqrt{z^3 + y} + y)\vec{k}$. Compute the flux of the vector field $\nabla \times F$ through a net whose rim is the unit circle $y^2 + z^2 = 1$ in the $x = 0$ plane, oriented in the $\vec{i}$ direction.

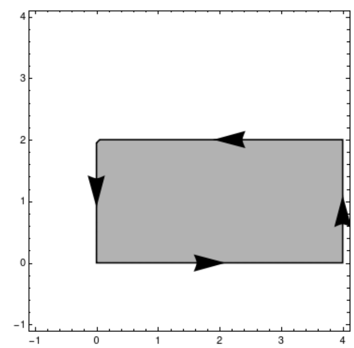



**Solution:** Instead of trying to parametrize the net, we use Stokes's theorem to just compute the circulation of $\vec{F}$ along the boundary. This means we don't even need to take the curl!

If the net is oriented in the $\vec{i}$ direction, that's the same as the circle being oriented counterclockwise when viewed from the positive $x$-axis. So we can parametrize the circle with $\vec{r}(t) = (0, \cos(t), \sin(t))$. Then by Stokes's theorem, we have

$$\int_S \nabla \times \vec{F} \cdot d\vec{A} = \int_C \vec{F} \cdot d\vec{r} = \int_0^{2\pi} (0, -\sin(t), \cos(t)) \cdot (0, -\sin(t), \cos(t))\, dt$$
$$= \int_0^{2\pi} \sin^2(t) + \cos^2(t)\, dt = \int_0^{2\pi} 1\, dt = 2\pi.$$

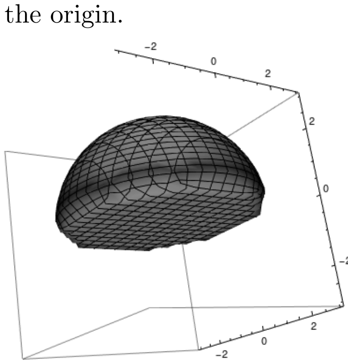
(b) Find the circulation of $\vec{F}(x, y) = -3y\vec{i} + 2x\vec{j}$ counterclockwise around the rectangle $0 \le x \le 4, 0 \le y \le 2$.




**Solution:** We compute $\|\nabla \times \vec{F}(x, y)\| = |2 - (-3)| = 5$. The curve is oriented so the interior is on the left-hand side, so by Green's Theorem, we have

$$\int_C \vec{F} \cdot d\vec{r} = \int_R \|\nabla \times \vec{F}\|\, dA = \int_0^2 \int_0^4 5\, dx\, dx = 40.$$

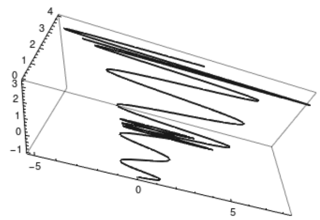
(c) Integrate the function $f(x, y, z) = z$ over the $z \ge 0$ half of the solid radius-3 spherical ball centered at the origin.



**Solution:** We use spherical coordinates. Then $f(r, \theta, \phi) = \rho \cos \phi$, and we have $0 \leq \rho \leq 3, 0 \leq \theta \leq 2\pi$, and $0 \leq \phi \leq \pi/2$. Thus we get

$$\begin{aligned}
\int_0^3 \int_0^{2\pi} \int_0^{\pi/2} \rho \cos \phi (\rho^2 \sin \phi) \, d\phi \, d\theta \, d\rho &= \int_0^3 \int_0^{2\pi} \int_0^{\pi/2} \rho^3 \cos \phi \sin \phi \, d\phi \, d\theta \, d\rho \\
&= \int_0^3 \int_0^{2\pi} \rho^3 \frac{1}{2} \sin^2 \phi |_0^{\pi/2} \, d\theta \, d\rho \\
&= \int_0^3 \int_0^{2\pi} \frac{\rho^3}{2} \, d\theta \, d\rho \\
&= \int_0^3 \rho^3 \pi \, d\rho = \frac{\pi}{4} \rho^4 |_0^3 = \frac{81\pi}{4}.
\end{aligned}$$

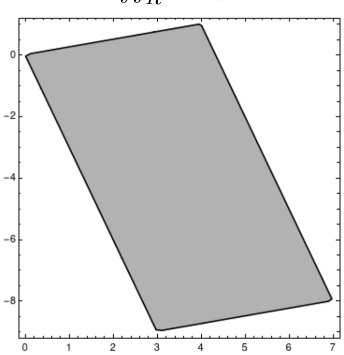
(d) Find the integral of the vector field $\vec{F}(x, y, z) = yz\vec{i} + xz\vec{j} + xy\vec{k}$ over the path $\vec{r}(t) = (t + \sin(10\pi t)e^t, t^2 - \cos(2\pi t), 2^t)$ as $t$ varies from 0 to 2.



**Solution:** We observe that $\vec{F}(x, y, z) = \nabla xyz$. Thus by the fundamental theorem of line integrals we can just plug in the two endpoints. We have

$$\begin{aligned}
\vec{r}(0) &= (0, -1, 1) \\
\vec{r}(2) &= (2, 3, 4) \\
\int_C \vec{F} \cdot \vec{r} &= 24 - 0 = 24.
\end{aligned}$$

(e) Compute $\iint_R x + y \, dA$ over the parallelogram with vertices $(0, 0), (4, 1), (7, -8), (3, -9)$.




**Solution:** We want to reparametrize this with $x = 4s + 3t, y = s - 9t$. [You could also use $x = 4s + t, y = s - 3t$, which would work out about the same.] Then we get bounds of $s, t \in [0, 1] \times [0, 1]$, and we're integrating the function $5s - 6t$.

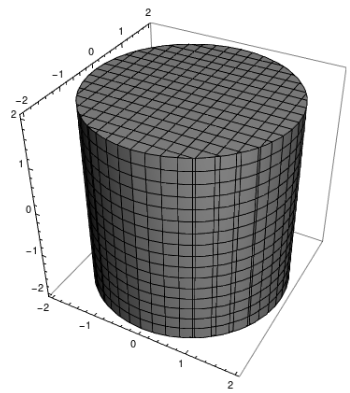
To compute the Jacobian get

$$\frac{\partial(x, y)}{\partial(s, t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} 4 & 3 \\ 1 & -9 \end{vmatrix} = -36 - 3 = -39$$

so $\left|\frac{\partial(x,y)}{\partial(s,t)}\right| = 39$. Then the integral is

$$\begin{aligned}\iint_R x + y\,dA &= \int_0^1 \int_0^1 5s - 6t \cdot 39\,dt\,ds \\ &= 39 \int_0^1 5st - 3t^2|_0^1\,ds = 39 \int_0^1 5s - 3\,ds \\ &= 39\left(5s^2/2 - 3s\right)\Big|_0^1 = 39(5/2 - 3) = -39/2.\end{aligned}$$

(f) Compute $\int_S \vec{F} \cdot d\vec{A}$, where $\vec{F}(x,y,z) = xy^2\vec{i} + x^2y\vec{j} + x^2y^2\vec{k}$ and $S$ is the surface (including both ends!) of a closed cylinder with radius 2 centered on the $z$-axis, from $z = -2$ to $z = 2$.




**Solution:** We want to use the divergence theorem here. We compute $\nabla \cdot \vec{F} = y^2 + x^2 + 0$, so we can integrate $x^2 + y^2$ over the cylinder. We use cylindrical coordinates, and get

$$\begin{aligned}\int_{-2}^{2} \int_0^{2\pi} \int_0^2 r^2 \cdot r\,dr\,d\,\theta\,dz &= \int_{-2}^{2} \int_0^{2\pi} \frac{1}{4} r^4|_0^2\,d\theta\,dz \\ &= \int_{-2}^{2} \int_0^{2\pi} 4\,d\theta\,dz = 32\pi.\end{aligned}$$