Math 2233 Practice Midterm 1 Solutions

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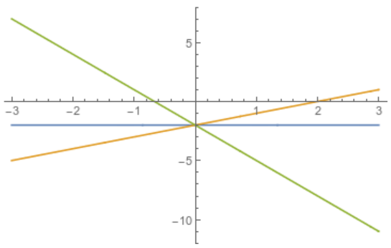
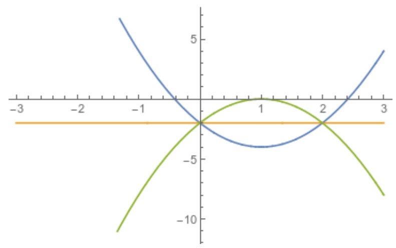
Problem 1. Let $f(x, y) = 2xy - x^2y - 2$

(a) Sketch cross-sections of f for $x = -1, 0, 1$ and $y = -2, 0, 2$.





Solution:

$$f(-1, y) = -2y - y - 2 = -3y - 2 \qquad f(0, y) = -2 \qquad f(1, y) = 2y - y - 2 = y - 2$$
$$f(x, -2) = -4x + 2x^2 - 2 \qquad f(x, 0) = -2 \qquad f(x, 2) = 4x - 2x^2 - 2$$

(b) If $\vec{u} = \frac{-3}{5}\vec{i} + \frac{4}{5}\vec{j}$, compute $f_{\vec{u}}(2, 1)$.

Solution: $\nabla f(x, y) = (2y - 2xy)\vec{i} + (2x - x^2)\vec{j}$, so

$$\nabla f(2, 1) = -2\vec{i}$$
$$f_{\vec{u}}(2, 1) = 6/5.$$

(c) At the point $(3, 1)$, in what direction should you move to increase $f(x, y)$ at the fastest possible rate? What is the rate of increase in that direction?

Solution: The direction of greatest increase is $\nabla f(3, 1) = -4\vec{i} - 3\vec{j}$. The rate of increase is $\|\nabla f(3, 1)\| = \sqrt{(-4)^2 + (-3)^2} = 5$.

Problem 2. (a) Give an equation for a plane through the points $(1, 1, 1), (1, 3, 5), (3, 1, -3)$.

Solution: There are two approaches here.

First, we can observe that the first two points share a x coordinate and the first and third share an y coordinate. Thus we can compute the x slope is -2 and the y slope is 2. Then our equation is

$$z = -2(x - 1) + 2(y - 1) + 1 = -2x + 2y + 1.$$

Alternatively, we get the vectors $2\vec{j} + 4\vec{k}$ and $2\vec{i} - 4\vec{k}$. Then we compute

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 2 & 4 \\ 2 & 0 & -4 \end{vmatrix} = -8\vec{i} + 8\vec{j} - 4\vec{k} = \vec{n}$$

and thus the equation for the plane is

$$0 = -8(x - 1) + 8(y - 1) - 4(z - 1).$$

These are, non-obviously, the same plane.

(b) Find the cosine of the angle between the vectors $\vec{v} = 3\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{u} = \vec{i} - 2\vec{j} + \vec{k}$.

Solution: We know that

$$\cos\theta = \frac{\vec{v}\cdot\vec{u}}{\|\vec{v}\|\cdot\|\vec{u}\|} = \frac{3-4-1}{\sqrt{14}\sqrt{6}} = \frac{-2}{2\sqrt{21}} = \frac{-1}{\sqrt{21}}.$$

(c) Let $\vec{v} = 3\vec{i} + \vec{j} - \vec{k}$ and $\vec{u} = -2\vec{i} - \vec{j} + 2\vec{k}$. Compute the orthogonal decomposition of $\vec{v}$ with respect to $\vec{u}$. That is, write $\vec{v} = \vec{v}_{\text{parallel}} + \vec{v}_{\perp}$.

Solution:

$$\begin{aligned} \vec{v}_{\text{parallel}} &= \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{-6-1-2}{4+1+4}\vec{u} \\ &= \frac{-9}{9}\vec{u} = 2\vec{i} + \vec{j} - 2\vec{k} \\ \vec{v}_{\perp} &= \vec{v} - \vec{v}_{\text{parallel}} = \vec{i} + \vec{k}. \end{aligned}$$

Problem 3. (a) Compute $\nabla\left(x^2z + \sqrt{xy}\right)$.

Solution:

$$(2xz + \frac{1}{2}\sqrt{y/x})\vec{i} + \frac{1}{2}\sqrt{x/y}\vec{j} + x^2\vec{k}$$

(b) Find an equation for the tangent plane to the graph of the function $f(x, y) = e^{xy} + x/y$ at the point $(0, 2)$.

Solution: We have $\nabla f(x,y) = (e^{xy}y + 1/y)\vec{i} + (e^{xy}x - x/y^2)\vec{j}$, so $\nabla f(0,2) = 5/2\vec{i} + 0\vec{j}$.

Further we have $f(0, 2) = 1 + 0$. Thus we get the equation

$$z = 1 + \frac{5}{2}(x - 0).$$

(c) Let $g(x, y, z) = x^2y + y^2z$. Use a linear approximation at the point $(1, 2, 3)$ to estimate $g(.9, 2.1, 3.2)$.
Solution:

$$\begin{aligned} \nabla g(x,y,z) &= 2xy\vec{i} + (x^2 + 2yz)\vec{j} + y^2\vec{k} \\ \nabla g(1,2,3) &= 4\vec{i} + 13\vec{j} + 4\vec{k} \\ g(x,y,z) &\approx 4(x-1) + 13(y-2) + 4(z-3) + 14 \\ g(.9, 2.1, 3.2) &\approx 4(-.1) + 13(.1) + 4(.2) + 14 = -.4 + 1.3 + .8 + 14 = 15.7. \end{aligned}$$

Problem 4. Let $f(x, y) = e^{xy-4} + x^2y$.

(a) Find the degree 2 Taylor polynomial for f centered at $(2, 2)$.

Solution:

We compute

$$\begin{aligned} f_x(x,y) &= ye^{xy-4} + 2xy & f_x(2,2) &= 10 \\ f_y(x,y) &= xe^{xy-4} + x^2 & f_y(2,2) &= 6 \\ f_{xx}(x,y) &= y^2e^{xy-4} + 2y & f_{xx}(2,2) &= 8 \\ f_{xy}(x,y) &= e^{xy-4} + xye^{xy-4} + 2x & f_{xy}(2,2) &= 9 \\ f_{yy}(x,y) &= x^2e^{xy-4} & f_{yy}(x,y) &= 4. \end{aligned}$$

Thus the Taylor polynomial is

$$T_2(x,y) = 9 + 10(x-2) + 6(y-2) + 4(x-2)^2 + 9(x-2)(y-2) + 2(y-2)^2.$$

(b) Use your answer in part (a) to estimate $f(1.9, 2.2)$.

Solution: We have

$$\begin{aligned} f(1.9, 2.2) &\approx 9 + 10(-.1) + 6(.2) + 4(-.1)^2 + 9(-.1)(.2) + 2(.2)^2 \\ &= 9 - 1 + 1.2 + .04 - .18 + .08 = 9.14. \end{aligned}$$

Problem 5. (a) Find and classify the critical points of $f(x, y) = 2x^3 + 6xy + 3y^2$.

Solution: We have

$$\begin{aligned} f_x(x, y) &= 6x^2 + 6y \\ f_y(x, y) &= 6x + 6y \end{aligned}$$

This gives us $y = -x$, and thus we have $x^2 - x = 0$ so x is either 0 or 1. Thus our critical points are $(0, 0)$ and $(1, -1)$.

We have

$$\begin{aligned} f_{xx}(x, y) &= 12x & f_{xx}(0, 0) &= 0 & f_{xx}(1, -1) &= 12 \\ f_{xy}(x, y) &= 6 & f_{xy}(0, 0) &= 6 & f_{xy}(1, -1) &= 6 \\ f_{yy}(x, y) &= 6 & f_{yy}(0, 0) &= 6 & f_{yy}(1, -1) &= 6. \end{aligned}$$

Then for $(0, 0)$ we have $D = 0 \cdot 6 - 6^2 = -36 < 0$, so we have a saddle point.

For $(1, -1)$ we have $D = 12 \cdot 6 - 6^2 = 36 > 0$, and $f_{xx}(1, -1) = 12 > 0$. So this is a local minimum.

(b) Find the critical points of $g(x, y, z) = 9x - 6x^2 + x^3 + x^2yz$.

Solution: We have

$$\begin{aligned} g_x(x, y, z) &= 9 - 12x + 3x^2 + 2xyz \\ g_y(x, y, z) &= x^2z \\ g_z(x, y, z) &= x^2y \end{aligned}$$

The third equation gives $x = 0$ or $y = 0$, and the second gives $x = 0$ or $z = 0$. If $x = 0$, then the first equation gives $1 = 0$ which is a contradiction, so we have $y = 0, z = 0$, and we solve $3x^2 - 12x + 9 = 0$.

Factoring this gives $0 = 3x^2 - 12x + 9 = 3(x^2 - 4x + 3) = 3(x - 3)(x - 1)$ so we have $x = 1$ or $x = 3$. Thus the two critical points are $(3, 0, 0)$ and $(1, 0, 0)$.