Math 2233 Practice Midterm 2 Solutions

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Problem 1. (a) Find the maximum and minimum values of $f(x, y) = 20 - 4x^2 - y^2$ on the disk $x^2 + y^2 \leq 4$.

Solution:

For interior critical points, we have $\nabla f(x, y) = (-8x, -2y)$, which gives the equations $-8x = 0$ and $-2y = 0$. Thus the only interior critical point is $(0, 0)$, and we compute $f(0, 0) = 20$.

On the boundary, we have

$$\begin{aligned} -8x &= 2\lambda x \\ -2y &= 2\lambda y. \end{aligned}$$

The second condition gives that $y = 0$ or $\lambda = -1$. If $y = 0$ then $x = \pm 2$; if $\lambda = -1$ then the first equation tells us that $x = 0$ and thus $y = \pm 2$. So we have four critical points:

$$\begin{aligned} f(2, 0) &= 4 \\ f(-2, 0) &= 4 \\ f(0, 2) &= 16 \\ f(0, -2) &= 16. \end{aligned}$$

So the absolute maximum on the disk is 20, and the absolute minimum is 4.

(b) Find a parametrization for the cone, opening in the direction of the x axis, with total inner angle $\pi/2$.

Solution:

$\vec{r}(s, t) = (\sqrt{s^2 + t^2}, s, t)$.

Alternatively, we can use cylindrical coordinates, and we have $\vec{r}(x, \theta) = (x, x\cos\theta, x\sin\theta)$.

Or we can use spherical coordinates, and get $\vec{r}(\rho, \theta) = (\rho\sqrt{2}/2, \rho\cos\theta\sqrt{2}/2, \rho\sin\theta\sqrt{2}/2)$.

Problem 2. (a) Find the volume of the region bounded by the planes $x = 3, x = 6 - y - z$, and $y = 0, z = 0$.

Solution: We have

$$\begin{aligned} V &= \int_0^3 \int_0^{3-y} \int_3^{6-y-z} 1\, dx\, dz\, dy \\ &= \int_0^3 \int_0^{3-y} 3 - y - z\, dz\, dy \\ &= \int_0^3 3z - yz - z^2/2|_0^{3-y}\, dy = \int_0^3 9 - 3y - (3y - y^2) - (9 - 6y + y^2)/2\, dy \\ &= \int_0^6 9/2 - 3y + y^2/2\, dy = 9y/2 - 3y^2/2 + y^3/6|_0^3 = -27/2 - 27/2 + 9/2 = 9/2. \end{aligned}$$

(b) Compute the integral of the function $f(x) = x + 3y$ over the region bounded by $x + 3y = 0, x + 3y = 3, x - 3y = 0, x - 3y = 2$. (Hint: reparametrize to get a rectangle).

Solution: We use $s = x + 3y, t = x - 3y$. Then we have $x = \frac{s+t}{2}$ and $y = \frac{s-t}{6}$. Then the Jacobian is

$$\frac{\partial(x, y)}{\partial(s, t)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/6 & -1/6 \end{vmatrix} = -1/12 - 1/12 = -1/6.$$

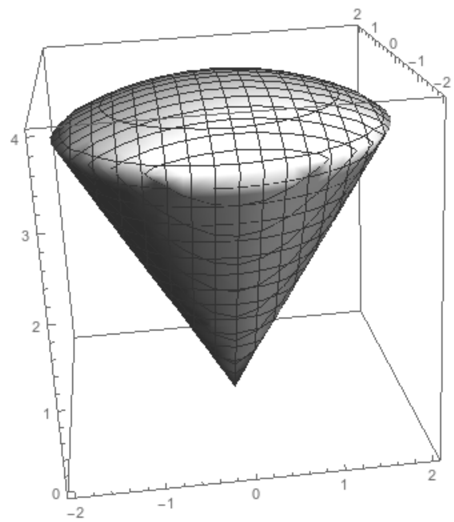
Thus our integral is

$$\int_0^3 \int_0^2 s|-1/6|\, dt\, ds = \int_0^3 \frac{s}{3}\, ds = \frac{s^2}{6}|_0^3 = \frac{3}{2}.$$

Problem 3. Let R be the spherical wedge bounded by a sphere of radius 4 centered at the origin, and the cone given by $z = \sqrt{3x^2 + 3y^2}$ (as shown below). Let $f(x, y, z) = z$.

(a) Set up integrals to compute $\int_R f\, dA$ in cartesian, cylindrical, and spherical coordinates.

(b) Choose one of these integrals and evaluate it.



Solution:

(a) We see that these intersect at the circle $x^2 + y^2 + 3x^2 + 3y^2 = 16$, or in other words $x^2 + y^2 = 4$, so the circle of radius 2 at the level $z = \sqrt{12} = 2\sqrt{3}$.

If we draw a triangle from the side, we see that we have a triangle with opposite side of length 2 and hypotenuse of length 4, so $\sin\phi = 1/2$. Thus $\phi = \pi/6$.

$$\int_{-2}^{2} \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{3x^2+3y^2}}^{\sqrt{16-x^2-y^2}} z\, dz\, dy\, dx$$

$$\int_0^{2\pi} \int_0^2 \int_{r\sqrt{3}}^{\sqrt{16-r^2}} zr\, dz\, dr\, d\theta$$

$$\int_0^4 \int_0^{2\pi} \int_0^{\pi/6} \rho\cos\phi\rho^2 \sin\phi\, d\phi\, d\theta\, d\rho$$

(b) I really hope everyone picks the spherical integral. We compute

$$\begin{aligned} I &= \int_0^4 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \cos\phi \sin\phi\, d\phi\, d\theta\, d\rho \\ &= \frac{1}{2} \int_0^4 \int_0^{2\pi} \rho^3 \sin^2\phi|_0^{\pi/6}\, d\theta\, d\rho \\ &= \frac{1}{8} \int_0^4 \int_0^{2\pi} \rho^3\, d\theta\, d\rho \\ &= \frac{\pi}{4} \int_0^4 \rho^3\, d\rho = \frac{\pi}{4}\frac{\rho^4}{4}|_0^4 = 16\pi. \end{aligned}$$

Problem 4. (a) Find a parametric equation for a particle moving in a straight line from $(1, 2, 2)$ to $(-3, 1, 5)$

Solution:

$$\vec{r}(t) = (1, 2, 2) + t(-4, -1, 3) = (1 - 4t, 2 - t, 2 + 3t).$$

(b) Suppose another particle follows the path $\vec{r}_2(t) = (t, t^2, 2 - t)$. Does this particle's path intersect the path of the particle from part (a)?

Solution:

We would need

$$\begin{aligned} 1 - 4t_1 &= t_2 \\ 2 - t_1 &= t_2^2 \\ 2 + 3t_1 &= 2 - t_2. \end{aligned}$$

The third equation gives that $t_2 = -3t_1$. Substituging into the first equation gives $1 - 4t_1 = -3t_1$ and thus $t_1 = 1$ (implying $t_2 = -3$). Then the second equation would give $2 - 1 = (-3)^2$, which is false. So the two paths never intersect.

(c) Find a flow line for the vector field $\vec{F}(x, y) = x\vec{i} + 1\vec{j}$ that goes through the point $(1, 1)$.

Solution: If $\vec{r}'(t) = \vec{F}(\vec{r}(t))$, we have $x'(t) = x(t)$ and $y'(t) = 1$, which means $x(t) = C_1 e^t$ and $y(t) = t + C_2$. If we assume this path goes through $(1, 1)$ at time $t = 0$ then we have $1 = C_1 \cdot e^0 = C_1$ and $1 = 0 + C_2$, so our final path is $\vec{r}(t) = (e^t, t + 1)$.