1 Vectors and Geometry

1.1 Vectors in Space

Definition 1.1 (informal)**.** A *vector* is a mathematical object that encodes both direction and size or *magnitude*. We notate vectors with an arrow over them, as in $\vec{v}$.

A *displacement vector* from one point to another is an arrow with its tail at one point and its head at the other. It gives the distance between the two points, and the direction from the first point to the second point. The vector from the point P to the point Q is written $\overrightarrow{PQ}$.

Notice that it is possible for $\overrightarrow{PQ}$ to be the same vector as $\overrightarrow{AB}$ even if A and B are different points from P and Q. The vector encodes the distance and direction, but not the specific points.

A displacement vector whose tail is at the origin is called a *position vector*.

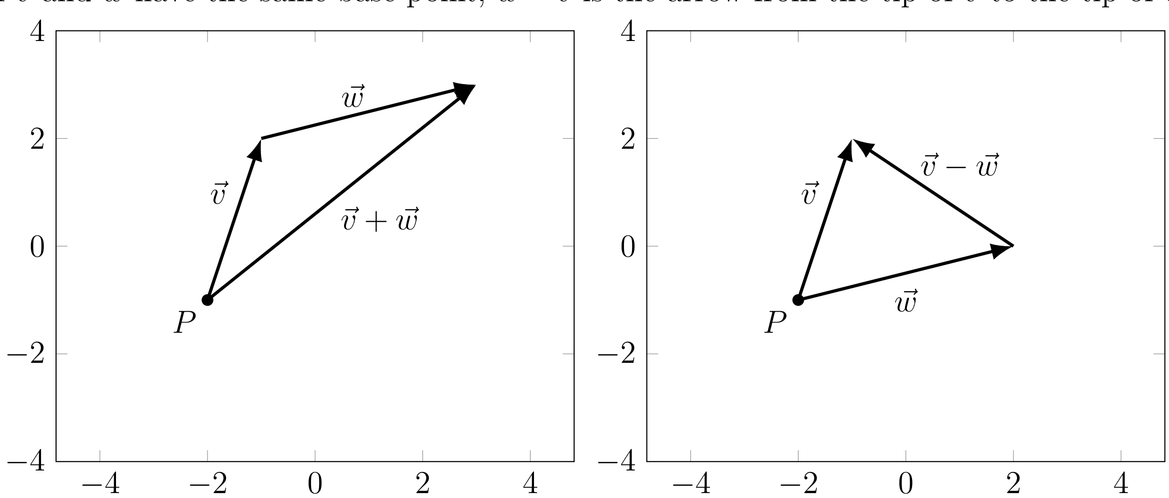
A quantity that has size but no direction is called a *scalar*.

Remark 1.2. We have a distinction between vectors and scalars in single-variable calculus, but we can mostly avoid thinking too hard about it since there are only two possible directions. Vectors show up, for instance, in the idea of one-sided limits.

We can do arithmetic on vectors. Adding vectors, geometrically, represents the displacement of following one vector and then the other, putting them tail-to-tip.

Definition 1.3. The *sum* $\vec{v} + \vec{w}$ of two vectors is the vector given by following $\vec{v}$ and then following $\vec{w}$. Thus if $\vec{v} = \overrightarrow{PQ}$ and $\vec{w} = \overrightarrow{QR}$ then $\vec{v} + \vec{w} = \overrightarrow{PR}$.

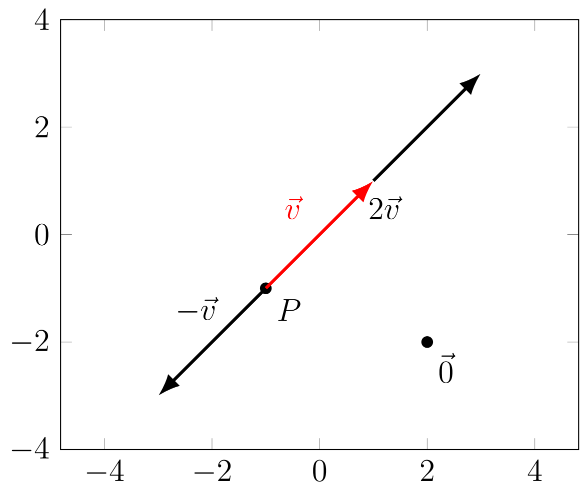
The *difference* of two vectors $\vec{w} - \vec{v}$ is the vector that, when added to $\vec{v}$, gives $\vec{w}$. That is, if $\vec{v}$ and $\vec{w}$ have the same base point, $\vec{w} - \vec{v}$ is the arrow from the tip of $\vec{v}$ to the tip of $\vec{w}$.



Scalar multiplication represents stretching a vector, and also possibly reversing its direction.

Definition 1.4. If λ is a scalar (real number), and $\vec{v}$ is a (displacement) vector, then the *scalar multiple* of $\vec{v}$ by λ is a vector stretched by a factor of $|\lambda|$. It points in the same direction as $\vec{v}$ if $\lambda > 0$ and in the opposite direction if $\lambda < 0$.

If $\lambda = 0$ then $\lambda\vec{v}$ is the *zero vector* $\vec{0}$, which has zero magnitude. This vector is the same regardless of direction, and corresponds to $\overrightarrow{PP}$ for any point P.



Vector arithmetic has a bunch of useful properties.

Fact 1.5. *Let $\vec{u}, \vec{v}, \vec{w}$ be vectors, and r, s be scalars. Then:*

1. *(Additive commutativity)* $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
2. *(Additive associativity)* $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$
3. *(Additive identity)* $\vec{u} + \vec{0} = \vec{u}$.
4. *(Additive inverses)* $\vec{u} + (-1)\vec{v} = \vec{u} - \vec{v}$. *We write* $(-1)\vec{v} = -\vec{v}$.
5. *(Distributivity)* $r(\vec{u} + \vec{v}) = r\vec{u} + r\vec{v}$
6. *(Distributivity)* $(r + s)\vec{u} = r\vec{u} + s\vec{u}$
7. *(Multiplicative associativity)* $r(s\vec{u}) = (rs)\vec{u}$
8. *(Multiplicative Identity)* $1\vec{u} = \vec{u}$
9. *(Zero Length)* $0\vec{v} = \vec{0}$.

Remark 1.6. In linear algebra we say this list of properties defines a *vector space*.

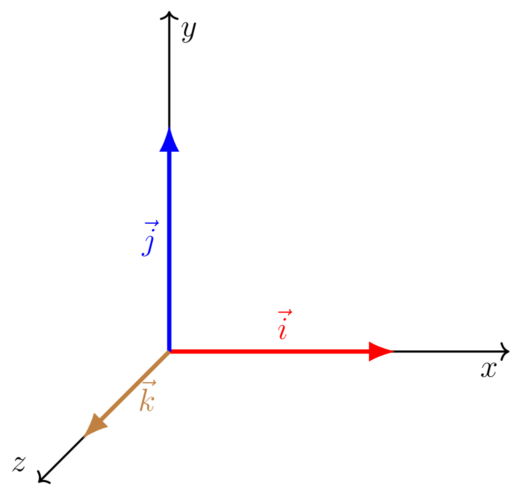
This seems like a long list of properties, but most of them are things you were probably assuming were true anyway. We could prove any of these properties by drawing vectors or working through some algebra, but we'll pass over that because it's boring and not very enlightening. .

1.2 Vector Components and Algebra

So far we've defined vectors and stated a bunch of properties they have, but all of this has been stated entirely in terms of geometry. We'd now like to establish the other side of our duality and express vectors in algebraic terms.

We first define three "standard" vectors.

Definition 1.7. We define the vector $\vec{i} = \overrightarrow{(0,0,0)(1,0,0)}$ to be the vector of length 1 in the positive x direction. Similarly we define $\vec{j}$ to be the vector of length 1 in the positive y direction, and $\vec{k}$ to be the vector of length 1 in the positive z direction.



Remark 1.8. In linear algebra terms, these three vectors are a *basis* for $\mathbb{R}^3$. In fact, these three vectors are the standard basis vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$.

We can take any vector in $\mathbb{R}^3$ and express it in terms of these vectors. If we know how far to go in the x direction, how far in the y direction, and how far in the z direction, then we know exactly where to go, so adding and multiplying these vectors can give us everything we need to identify any possible vector.

Definition 1.9. If $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ where the v_i are scalars, then we say that we have *resolved* $\vec{v}$ into components, and the summands are the *components* of $\vec{v}$.

We will sometimes write $\vec{v} = (v_1, v_2, v_3)$ or $\vec{v} = \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}$ or $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix}$. But in this course we will usually find it more convenient to use the $\vec{i}, \vec{j}, \vec{k}$ notation.

Example 1.10. Let $P = (1, 3, 2)$ and $Q = (2, -4, 1)$. Then the vector $\overrightarrow{PQ} = \vec{i} - 7\vec{j} - \vec{k}$.

The vector $2\vec{i} - 14\vec{j} - 2\vec{k}$ is parallel to $\overrightarrow{PQ}$ since it is a scalar multiple of $\overrightarrow{PQ}$. But the vector $\vec{i} + 7\vec{j} + \vec{k}$ is not.

This sort of resolution or decomposition makes it easy to do vector arithmetic algebraically.

Example 1.11. If $\vec{v} = \overrightarrow{PQ} = \vec{i} - 7\vec{j} - \vec{k}$ and $\vec{u} = 2\vec{i} - 3\vec{j} + 2\vec{k}$, then $\vec{u} + \vec{v} = 3\vec{i} - 10\vec{j} + \vec{k}$.

We have $3\vec{v} = 3\vec{i} - 21\vec{j} - 3\vec{k}$.

1.3 Angles, Magnitudes and the Dot Product

We talk about vectors as having a direction and a magnitude. We've talked about this geometrically in terms of arrows; we've algebraically resolved them into components. But we can also numerically specify a direction in terms of angles, and then simply give a magnitude.

1.3.1 Vector Magnitudes

Definition 1.12. The *magnitude* of a vector $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ is $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + v_2^2 + v_3^2}$.

If $\|\vec{v}\| = 1$ then we say that $\vec{v}$ is a *unit vector*.

Example 1.13. Let $\vec{v} = 3\vec{i} + 1\vec{j} - 2\vec{k}$ and $\vec{w} = 2\vec{i} - \vec{j} + 4\vec{k}$. Then $\|\vec{v}\| = \sqrt{3^2 + 1^1 + 2^2} = \sqrt{14}$ and $\|\vec{w}\| = \sqrt{2^2 + 1^2 + 4^2} = \sqrt{21}$.

If we want unit vectors in the same direction, we can take the vectors

$$\frac{\vec{v}}{\|\vec{v}\|} = \frac{\vec{v}}{\sqrt{14}} = \frac{3}{\sqrt{14}}\vec{i} + \frac{1}{\sqrt{14}}\vec{j} - 2\frac{2}{\sqrt{14}}\vec{k}$$
$$\frac{\vec{w}}{\|\vec{w}\|} = \frac{\vec{w}}{\sqrt{21}} = \frac{2}{\sqrt{21}}\vec{i} - \frac{1}{\sqrt{21}}\vec{j} + \frac{4}{\sqrt{21}}\vec{k}.$$

Example 1.14. Let's find a unit vector based at the point (x, y, z) that points directly away from the origin.

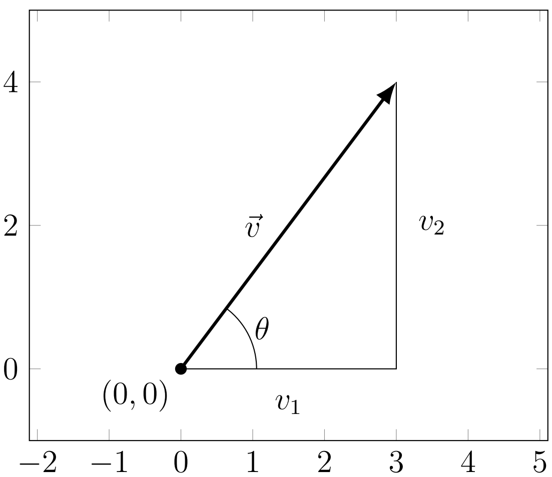
We need a vector that points in the same direction as $x\vec{i} + y\vec{j} + z\vec{k}$, but has unit length. So we compute

$$\|x\vec{i} + y\vec{j} + z\vec{k}\| = \sqrt{x^2 + y^2 + z^2}$$

so our unit vector is

$$\frac{x}{\sqrt{x^2+y^2+x^2}}\vec{i}+\frac{y}{\sqrt{x^2+y^2+x^2}}\vec{j}+\frac{z}{\sqrt{x^2+y^2+x^2}}\vec{k}.$$

From the Pythagorean theorem we can see that the magnitude is actually the length of the arrow corresponding to the vector. In two dimensions, we see that the length of the vector $v_1\vec{i}+v_2\vec{j}$ is $\sqrt{v_1^2+v_2^2}$; the same argument works in three (or more) dimensions by repeatedly applying the Pythagorean theorem.



We can also use trigonometry to relate the components v_1 and v_2 with the magnitude $\|\vec{v}\|$ and the angle θ.

Proposition 1.15. *If a vector $\vec{v}=v_1\vec{i}+v_2\vec{j}$ makes an angle of θ with the x axis, then:*

- $v_1=\|\vec{v}\|\cos\theta$
- $v_2=\|\vec{v}\|\sin\theta$
- $\theta=\arctan {}^{v_2}/_{v_1}$.

Example 1.16. Let $\vec{v}=3\vec{i}+4\vec{j}$. Then $\|\vec{v}\|=\sqrt{3^2+4^2}=5$, and $\vec{v}$ makes an angle of approximately .92 with the x axist.

Let $\vec{w}$ have length 7 and make an angle of $\pi/6$ with the x axis. Then

$$\vec{w}=7\cos(\pi/6)\vec{i}+7\sin(\pi/6)\vec{j}=\frac{7\sqrt{3}}{2}\vec{i}+\frac{7}{2}\vec{j}\approx 6.06\vec{i}+3.5\vec{j}.$$

1.3.2 The Dot Product

To think more about angles and magnitudes, we want to consider the dot product.

Definition 1.17. If $\vec{v} = v_1\vec{i} + v_2\vec{j} + v_3\vec{k}$ and $\vec{w} = w_1\vec{i} + w_2\vec{j} + w_3\vec{k}$ then the *dot product* $\vec{v} \cdot \vec{w} = v_1w_1 + v_2w_2 + v_3w_3$.

Example 1.18. Let $\vec{v} = 3\vec{i} + 1\vec{j} - 2\vec{k}$ and $\vec{w} = 2\vec{i} - \vec{j} + 4\vec{k}$. Then $\vec{v} \cdot \vec{w} = 6 - 1 - 8 = -3$.

Notice the dot product takes in two vectors and gives a scalar. Also notice that $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$.

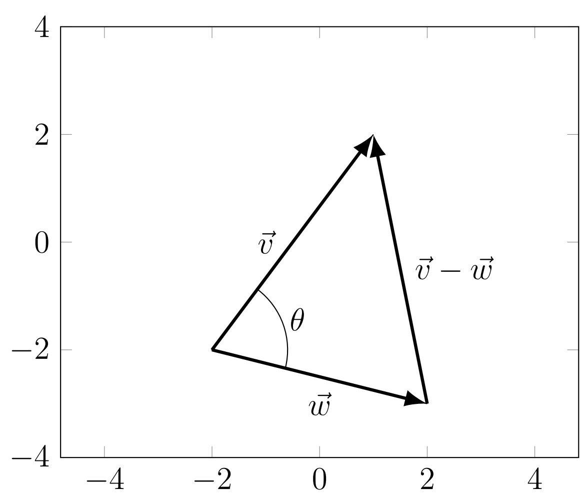
Proposition 1.19. *If $\vec{u}, \vec{v}, \vec{w}$ are vectors and λ is a scalar, then:*

- $\vec{v} \cdot \vec{w} = \vec{w} \cdot \vec{v}$
- $\vec{v} \cdot (\lambda\vec{w}) = \lambda(\vec{v} \cdot \vec{w}) = (\lambda\vec{v}) \cdot \vec{w}$
- $(\vec{v} + \vec{w}) \cdot \vec{u} = \vec{v} \cdot \vec{u} + \vec{w} \cdot \vec{u}$.

We can use the dot product to find the angles between two vectors.

Proposition 1.20. *If $\vec{v}, \vec{w}$ are vectors, and θ is the angle between them, then $\vec{v} \cdot \vec{w} = \|\vec{v}\|\|\vec{w}\| \cos\theta$.*

Proof. We can prove this using the law of cosines, which states that if a triangle has sides of length a, b, c, and the angle opposite side c has measure θ, then $c^2 = a^2 + b^2 - 2ab\cos\theta$. (Notice that if c is the hypotenuse of a right triangle, then $\cos\theta = \cos\pi/2 = 0$ and we recover the pythagorean theorem).



So form a triangle with sides $\vec{v}, \vec{w}$, and $\vec{v} - \vec{w}$. Then the law of cosines gives us

$$\begin{aligned} \|\vec{v}\|^2 + \|\vec{w}\|^2 - 2\|\vec{v}\|\|\vec{w}\|\cos\theta &= \|\vec{v} - \vec{w}\|^2 \\ \left(v_1^2 + v_2^2 + v_3^2\right) + \left(w_1^2 + w_2^2 + w_3^2\right) - 2\|\vec{v}\|\|\vec{w}\|\cos\theta &= (v_1 - w_1)^2 + (v_2 - w_2)^2 + (v_3 - w_3)^2 \\ &= v_1^2 - 2v_1w_1 + w_1^2 \\ &\quad + v_2^2 - 2v_2w_2 + w_2^2 \\ &\quad + v_3^2 - 2v_3w_3 + w_3^2 \end{aligned}$$

and subtracting the squared terms from both sides gives

$$\begin{aligned} -2\|\vec{v}\|\|\vec{w}\|\cos\theta &= -2v_1w_1 - 2v_2w_2 - 2v_3w_3 \\ \|\vec{v}\|\|\vec{w}\|\cos\theta &= v_1w_1 + v_2w_2 + v_3w_3 = \vec{v} \cdot \vec{w}. \end{aligned}$$

□

Example 1.21. Suppose $\|\vec{v}\| = 3$ and $\|\vec{w}\| = 5$. What is the maximum possible dot product, and when does this happen? What is the minimum possible dot product, and when does this happen? When is the dot product 0?

The dot product is $\vec{v} \cdot \vec{w} = 15\cos\theta$. So the dot product is maximized when $\cos\theta$ is maximized, which happens when $\theta = 0$; thus when the vectors point in the same direction, their dot product is 15. Similarly, when $\theta = \pi$ the vectors point in opposite directions, and their dot product is minimized at -15.

The dot product is 0 when $\cos\theta = 0$, which happens when $\theta = \pi/2$. Thus this occurs when the vectors are at a right angle.

Definition 1.22. We say that two vectors $\vec{v}$ and $\vec{w}$ are *perpendicular* or *orthogonal* if $\vec{v} \cdot \vec{w} = 0$. This corresponds to the two vectors forming a right angle.

If we know the angle between two vectors, we can use this to compute the dot product; but more often we use the dot product to compute the angle between two vectors.

Example 1.23. Let $\vec{v} = 3\vec{i} + 2\vec{j}$ and $\vec{w} = 2\vec{i} - \vec{j}$. Then $\vec{v} \cdot \vec{w} = 6 - 2 = 4$. We see that $\|\vec{v}\| = \sqrt{13}$ and $\|\vec{w}\| = \sqrt{5}$ so

$$\theta = \arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right) = \arccos\left(\frac{4}{\sqrt{65}}\right) \approx 1.05.$$

Let $\vec{v} = 3\vec{i} + 2\vec{j} - \vec{k}$ and $\vec{w} = 2\vec{i} - \vec{j} + 4\vec{k}$. Then $\vec{v} \cdot \vec{w} = 6 - 2 - 4 = 0$. Thus the angle between $\vec{v}$ and $\vec{w}$ is

$$\arccos\left(\frac{\vec{v} \cdot \vec{w}}{\|\vec{v}\|\|\vec{w}\|}\right) = \arccos(0) = \pi/2.$$

We see that $\vec{v}$ and $\vec{w}$ are orthogonal.

1.3.3 The Dot Product, Lines, and Planes

We can use the dot product and angles to understand lines and planes better.

First we consider a plane. We can think of a given plane as being the set of all lines thought a given point perpendicular to a given line. We want to rephrase this idea in terms of vectors.

Definition 1.24. If $\vec{n}$ is perpendicular to a plane—that is, perpendicular to any vector between two points in the plane—then we say that $\vec{n}$ is a *normal vector* to the plane.

If $P = (x_0, y_0, z_0)$ is a point on the plane, then the plane consists of all points $Q = (x, y, z)$ such that $\overrightarrow{PQ}$ is perpendicular to $\vec{n} = a\vec{i} + b\vec{j} + c\vec{k}$. Since $\overrightarrow{PQ} = (x - x_0)\vec{i} + (y - y_0)\vec{j} + (z - z_0)\vec{k}$, we see that the plane is the set of points satisfying

$$\vec{n} \cdot \overrightarrow{PQ} = 0$$
$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Notice that this is equivalent to the equation for a plane we gave in section **??**. In particular, if we take $d = ax_0 + by_0 + cz_0$ then our equation is

$$ax + by + cz = d.$$

Example 1.25. Find an equation for the plane perpendicular to $\vec{n} = \vec{i} - 2\vec{j} + \vec{k}$ and passing through the point $(1, 5, 2)$.

The equation is

$$(x - 1) - 2(x - 5) + (x - 2) = 0.$$

Example 1.26. Find a normal vector to the plane given by the equation

$$5x - 2y + 3z = 32.$$

$\vec{n} = 5\vec{i} - 2\vec{j} + 3\vec{k}$.

Find a normal vector to the plane given by

$$z = 3(x - 1) + 2(y - 5) + 2.$$

We rewrite the equation to be $0 = 3(x - 1) + 2(y - 5) - z + 2$. Then we see that $\vec{n} = 3\vec{i} + 2\vec{j} - \vec{k}$.

We can also use the dot product to project a vector onto a line. In fact this is probably the best way to understand what the dot product "really means".

Earlier we used trigonometry to resolve a vector into its $\vec{i}$ and $\vec{j}$ components. But we can do the same thing to a coordinate axis defined by any vector.

Definition 1.27. Let $\vec{v}, \vec{u}$ be two non-zero vectors. We define the *projection* of $\vec{v}$ onto $\vec{u}$ to be

$$\vec{v}_{\text{parallel}} = \frac{\vec{v} \cdot \vec{u}}{\|\vec{u}\|^2}\vec{u} = \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\vec{u}.$$

In particular, if $\vec{u}$ is a unit vector, then $\vec{v}_{\text{parallel}} = (\vec{v} \cdot \vec{u})\vec{u}$.

We define the *orthogonal complement* of $\vec{v}$ on $\vec{u}$ to be $\vec{v}_{\text{perp}} = \vec{v}_{\perp} = \vec{v} - \vec{v}_{\text{parallel}}$.

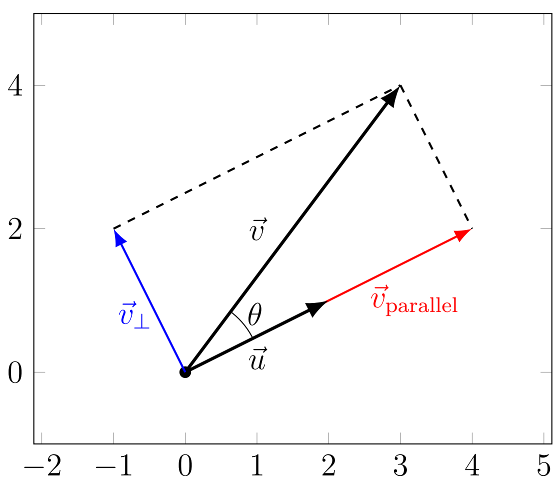
Proposition 1.28. *Let $\vec{v}, \vec{u}$ be non-zero vectors. Then:*

- *$\vec{v}_{parallel}$ is parallel to $\vec{u}$.*
- *$\vec{v}_{\perp} \cdot \vec{u} = 0$.*

Thus $\vec{v}_{parallel}$ represents the part of $\vec{v}$ that goes in the same direction as $\vec{u}$, and $\vec{v}_{\perp}$ represents the remainder.

Proof. We'll prove the first fact assuming the angle θ between $\vec{v}$ and $\vec{u}$ satisfies $0 \leq \theta \leq \pi/2$. The proof for $\pi/2 \leq \theta \leq \pi$ is basically the same but involves a slightly different picture. We never need to consider $\theta > \pi$ since we can just look at the angle going around the other way.

Consider the following diagram:



We see by trigonometry that $\|\vec{v}_{\text{parallel}}\| = \|\vec{v}\| \cos\theta$. Since we know that $\vec{v}_{\text{parallel}}$ has the same direction as $\vec{u}$, we see that

$$\begin{aligned}
\vec{v}_{\text{parallel}} &= \|\vec{v}\| \cos\theta \frac{\vec{u}}{\|\vec{u}\|} \\
&= \|\vec{v}\|\|\vec{u}\| \cos\theta \frac{1}{\|\vec{u}\|^2}\vec{u} \\
&= \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}
\end{aligned}$$

by proposition **??**.

We can now prove the second fact with simple algebra. We have

$$\begin{aligned}
\vec{v}_{\perp}\cdot\vec{u} = (\vec{v} - \vec{v}_{\text{parallel}})\cdot\vec{u} &= \vec{v}\cdot\vec{u} - \vec{v}_{\text{parallel}}\cdot\vec{u} \\
&= \vec{v}\cdot\vec{u} - \left(\frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}\right)\cdot\vec{u} \\
&= \vec{v}\cdot\vec{u} - \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}(\vec{u}\cdot\vec{u}) = \vec{v}\cdot\vec{u} - \vec{v}\cdot\vec{u} = 0.
\end{aligned}$$

□

Example 1.29. Let's decompose the vector $\vec{v} = 3\vec{i} + 4\vec{j}$ with respect to $\vec{u} = 2\vec{i} + \vec{j}$.

We compute

$$\begin{aligned}
\vec{v}_{\text{parallel}} &= \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = \frac{6+4}{4+1}(2\vec{i}+\vec{j}) = 4\vec{i} + 2\vec{j} \\
\vec{v}_{\perp} &= \vec{v} - \vec{v}_{\text{parallel}} = 3\vec{i} + 4\vec{j} - 4\vec{i} - 2\vec{j} = -\vec{i} + 2\vec{j}.
\end{aligned}$$

We can see that $\vec{v}_{\text{parallel}}$ is indeed parallel to $\vec{u}$, and that $\vec{v}_{\perp} + \vec{v}_{\text{parallel}} = \vec{v}$. We also check that

$$\vec{v}_{\perp}\cdot\vec{u} = (-\vec{i} + 2\vec{j})\cdot(2\vec{i}+\vec{j}) = -2 + 2 = 0.$$

Thus $\vec{v}_{\perp} \perp \vec{u}$ as we wanted.

Example 1.30. Let's decompose $\vec{v} = 2\vec{i} - 3\vec{j} + \vec{k}$ onto $\vec{u} = \vec{i} + \vec{j} - 2\vec{k}$.

$$\begin{aligned}
\vec{v}_{\text{parallel}} &= \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} = 2 - 3 - 21 + 1 + 4(\vec{i} + \vec{j} - 2\vec{k}) = -\vec{i}/2 - \vec{j}/2 + \vec{k} \\
\vec{v}_{\perp} &= \vec{v} - \vec{v}_{\text{parallel}} = 2\vec{i} - 3\vec{j} + \vec{k} + \vec{i}/2 + \vec{j}/2 - \vec{k} = 5\vec{i}/2 - 5\vec{j}/2.
\end{aligned}$$

And we check that

$$\vec{u}\cdot\vec{v}_{\perp} = (\vec{i} + \vec{j} - 2\vec{k})\cdot(5\vec{i}/2 - 5\vec{j}/2) = 5/2 - 5/2 = 0.$$

One common application of this idea is *work* in physics. The work done by a force on an object is the force applied in the direction of motion, times the distance traveled. But if the force is not in the direction of motion (which happens, for instance, if the object is already moving when force is applied), then we can use the dot product to calculate the work.

Example 1.31. Suppose a force of 10 Newtons is applied in the $\vec{i}+\vec{j}$ direction to an object that moves along displacement $\vec{d} = 3\vec{i} - 2\vec{j}$. What is the work done?

We can decompose the force into the force into the force $\vec{F}_{\text{parallel}}$ in the direction of the displacement, and the force $F_\perp$ perpendicular to the direction.

But in fact we know that $\|\vec{F}_{\text{parallel}}\| = \|\vec{F}\| \cos\theta$, so

$$W = \|\vec{F}_{\text{parallel}}\|\|\vec{d}\| = \|\vec{F}\| \cos\theta \|\vec{d}\| = \vec{F} \cdot \vec{d}.$$

Thus we can use the dot product to compute the work. So we have

$$W = \vec{F} \cdot \vec{d} = 15\sqrt{2} - 10\sqrt{2} = 5\sqrt{2}.$$

This concept will come up a lot when we start studying integrals.

1.4 The Cross Product

The dot product takes in two vectors and gives us a scalar. But often we want to take two vectors and get another vector. The cross product answers a very specific question: given two vectors, find a vector that is perpendicular to both of them.

Since there are two directions "perpendicular to both vectors", we choose one based on the *right-hand rule*, which says that if you point the fingers of your right hand towards $\vec{u}$ and curl your fingers towards $\vec{v}$ then your thumb will point in the direction of $\vec{u} \times \vec{v}$.

Definition 1.32. Let $\vec{u}$ and $\vec{v}$ be vectors, and let θ be the angle between them. Then the *cross product* of $\vec{u}$ and $\vec{v}$ is given by

$$\vec{u} \times \vec{v} = (\|\vec{u}\|\|\vec{v}\| \sin\theta)\vec{n}$$

where $\vec{n}$ is a unit normal vector to the plane containing $\vec{u}$ and $\vec{v}$, pointing in the direction given by the right-hand rule.

Remark 1.33. If $\vec{u}$ and $\vec{v}$ point in the same direction, then there is more than one plane through both of them and the vector $\vec{n}$ isn't clearly defined. But in this case $\sin\theta = \sin(0) = 0$ so $\vec{u} \times \vec{v} = \vec{0}$, and we don't need to worry about the direction.

Fact 1.34. *let $\vec{u}$ and $\vec{v}$ be vectors. Then*

$$\vec{u} \times \vec{v} = (u_2v_3 - u_3v_2)\vec{i} + (u_3v_1 - u_1v_3)\vec{j} + (u_1v_2 - u_2v_1)\vec{k}.$$

We often remember this definition by writing it as a determinant:

$$\vec{u} \times \vec{v} = \det \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{bmatrix} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}.$$

Example 1.35. Let $\vec{u} = \vec{i} - 3\vec{j} + \vec{k}$ and $\vec{v} = -2\vec{i} + \vec{j} + 2\vec{k}$. Then

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & -3 & 1 \\ -2 & 1 & 2 \end{vmatrix} = (-6-1)\vec{i} + (-2-2)\vec{j} + (1-6)\vec{k} = -7\vec{i} - 4\vec{j} - 5\vec{k}.$$

We can check that this is perpendicular to both $\vec{u}$ and $\vec{v}$.

The cross product makes it really easy to find equations for planes in threespace. If you have two vectors parallel to the plane, the cross product will give you a normal vector, which is all you need to write down the equation for the plane.

Example 1.36. Find an equation for the plane parallel to the vectors $\vec{u} = \vec{i} + \vec{j}$ and $\vec{v} = 5\vec{i} - \vec{j} + 3\vec{k}$ through the point $(5, 1, 3)$.

We compute

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 0 \\ 5 & -1 & 3 \end{vmatrix} = (3-0)\vec{i} + (0-3)\vec{j} + (-1-5)\vec{k} = 3\vec{i} - 3\vec{j} - 6\vec{k}.$$

We can double-check that this is perpendicular to both our original vectors.

For the equation of the plane we need a point, which we have, and a normal vector, which is $\vec{u} \times \vec{v}$. Thus the equation for the plane is

$$0 = 3(x-5) - 3(y-1) - 6(z-3).$$

Example 1.37. Use the cross product to find an equation for the plane containing the three points $(1, 4, 2), (5, 1, 1), (-2, 1, 7)$.

We see that the vectors from the first point to the other two are $4\vec{i}-3\vec{j}-\vec{k}$ and $-3\vec{i}-3\vec{j}+5\vec{k}$. Thus we compute

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -3 & -1 \\ -3 & -3 & 5 \end{vmatrix} = (-15-3)\vec{i} + (3-20)\vec{j} + (-12-9)\vec{k} = -18\vec{i} - 17\vec{j} - 21\vec{k}.$$

Thus an equation for the plane is

$$0 = 18(x-1) + 17(y-4) + 21(z-2).$$

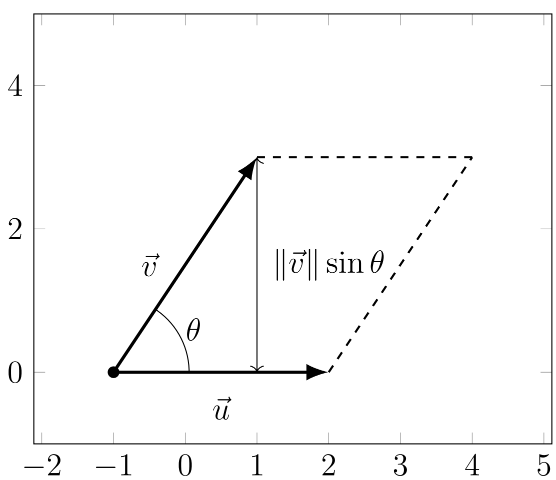
Proposition 1.38. *If $\vec{u}, \vec{v}, \vec{w}$ are vectors and λ is a scalar, then:*

- $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$.
- $(\lambda\vec{u}) \times \vec{v} = \lambda(\vec{u} \times \vec{v}) = \vec{y} \times (\lambda\vec{v})$.
- $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$.

We can also interpret the cross product as measuring an area.

Proposition 1.39. *Let $\vec{u}, \vec{v}$ be two vectors in a plane, and let θ be the angle between them. Then the area of the parallelogram with $\vec{u}$ and $\vec{v}$ as two sides is $\|\vec{u} \times \vec{v}\|$.*

Proof. The area of a parallelogram is the length of the base times the height.



The length of the base is $\|\vec{u}\|$, and the height is $\|\vec{v}\| \sin\theta$ by basic trigonometry. Thus the area of the parallelogram is $\|\vec{v}\|\|\vec{u}\| \sin\theta = \|\vec{v} \times \vec{u}\|$.

□

Example 1.40. Find the area of the parallelogram with corners at $(-1, 0), (2, 0), (1, 3), (4, 3)$.

We see that two of the vectors here are $3\vec{i}$ and $2\vec{i} + 3\vec{j}$. Thus we have

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & 0 \\ 2 & 3 & 0 \end{vmatrix} = 0\vec{i} + 0\vec{j} + (9-0)\vec{k} = 9\vec{k}$$

so $\|\vec{u} \times \vec{v}\| = 9$.