Math 2233 Fall 2021
Multivariable Calculus Mastery Quiz 10
Due Thursday, December 9

This week's mastery quiz has four topics. **Submit no more than three**. If you already have a 2/2 on a topic, you should not submit it.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in at class/recitation on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

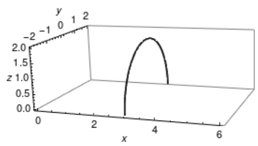
- Topic 11: Line Integrals
- Topic 12: Conservative Vector Fields
- Topic 13: Surface Integrals
- Topic 14: Green's and Stokes's

Name:

Recitation Section:

Topic 11: Line Integrals

(a) Set up, but **do not evaluate**, an integral to compute: the mass of a wire following a semi-circular path of radius 2 contained in the $x = 3$ plane, which goes from $(3, 2, 0)$ through $(3, 0, -2)$ to $(3, -2, 0)$, with density given by $\delta(x, y, z) = x^2 + y^2 + \sqrt{z^2 + 1}$?



Solution: We can parametrize with $\vec{r}(t) = (3, 2\cos t, -2\sin t)$. Then the integral is

$$\int_0^{\pi} (3^2 + 4\cos^2 t + \sqrt{4\sin^2 t + 1})\sqrt{4\sin^2 t + 4\cos^2 t}\, dt = \int_0^{\pi} 2(9 + 4\cos^2 t + \sqrt{4\sin^2 t + 1})\, dt.$$

(b) Let C be the curve $y = x^2$ from $(0, 0)$ to $(1, 1)$.

Compute the line integral of the vector field $\vec{F}(x, y) = (xy, -x^2)$.

Solution:

We parametrize the curve with $\vec{r}(t) = (t, t^2)$. Then we have

$$\int \vec{F} \cdot d\vec{r} = \int_0^1 (t^3, -t^2) \cdot (1, 2t)\, dt = \int_0^1 t^3 - 2t^3\, dt$$
$$= \int_0^1 -t^3\, dt = \frac{-t^4}{4}\Big|_0^1 = -1/4.$$

Topic 12: Conservative Vector Fields

(a) Find a potential field for $\vec{F}(x, y, z) = (y + z)\vec{i} + (x + z^2)\vec{j} + (x + 2yz)\vec{k}$ or prove none exists.

Solution:

If $\vec{F} = \nabla f$, we must have

$$f(x, y, z) = xy + xz + g_1(y, z)$$
$$f(x, y, z) = xy + yz^2 + g_2(x, z)$$
$$f(x, y, z) = xz + yz^2 + g_3(x, y)$$

We can satisfy all of these requirements with $f(x, y, z) = xy + xz + yz^2$.

(b) Let $f(x, y, z) = x^3y - xz^2$. Compute $\int_C \nabla f\, ds$ where C is parametrized by the curve $\vec{r}(t) = (t + 1, t^2 - 2, \sin(\pi t))$ for $t \in [0, 2]$.

Solution:

We could compute the whole line integral out, but that would be awful. Instead we use the fundamental theorem of line integrals, and we have

$$\begin{aligned}\int_C \nabla f\, ds &= f(r(2)) - f(r(0)) \\ &= f((3,2,0) - f((1,-2,0)) \\ &= 54 - 0 - (-2) + 0 = 56.\end{aligned}$$

Topic 13: Surface Integrals

(a) The moment of inertia of a surface about the z axis is given by the formula $I = \iint_S x^2 + y^2\, dS$. Find the moment of inertia of the surface $z = xy$ lying inside the cylinder $x^2 + y^2 = 3$.

Solution:

We want to compute $\iint_S x^2 + y^2\, dS = \iint_R (x^2 + y^2)\|\vec{r}_s \times \vec{r}_t\|\, ds\, dt$. We know from class that if our surface is the graph of a function, then

$$\|\vec{r}_s \times \vec{r}_t\| = \sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + y^2 + x^2}.$$

We're integrating over the circle $x^2 + y^2 = 3$, so we should use polar coordinates. So we get the integral

$$\int_0^{\sqrt{3}} \int_0^{2\pi} r^2\sqrt{1 + r^2} r\, d\theta\, dr = \int_0^{\sqrt{3}} 2\pi r^3 \sqrt{1 + r^2}\, dr$$

We now do the substitution $u = 1 + r^2$ and get

$$\begin{aligned}\int_0^{\sqrt{3}} 2\pi r^3 \sqrt{1 + r^2}\, dr &= \pi \int_1^4 (u - 1)\sqrt{u}\, du = \pi \int_1^4 u^{3/2} - u^{1/2}\, du \\ &= \pi \left(\frac{2u^{5/2}}{5} - \frac{2u^{3/2}}{3} \right)\Bigg|_1^4 \\ &= \pi \left(\frac{64}{5} - \frac{16}{3} - \frac{2}{5} + \frac{2}{3} \right) = \frac{116}{15}\pi.\end{aligned}$$

(b) Let V be the volume between the spherical shells of radius 1 and 2 centered at the origin. Compute the flux of the vector field $\vec{F}(x, y, z) = x\vec{i} + y\vec{j}$ out of the volume V.

Solution:

We have to compute two fluxes: out of the sphere of radius 2 oriented outwards, and out of the sphere of radius 1 oriented inwards. Let's look at the outer sphere: we can parametrize it in spherical coordinates, and using our formula we have

$$\begin{aligned}
\text{Flux} &= \int_0^{2\pi}\int_0^{\pi} (2\sin(\phi)\cos(\theta), 2\sin(\phi)\sin(\theta), 0)\cdot(\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi) 4\sin(\phi)\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} 8\sin^3\phi\cos^2\theta + 8\sin^3\phi\sin^2\theta\, d\phi\, d\theta \\
&= \int_0^{\pi}\int_0^{2\pi} 8\sin^3\phi\, d\theta\, d\phi \\
&= \int_0^{\pi} 16\pi\sin^3(\phi)\, d\phi = 16\pi\int_0^{\pi} \sin(\phi) - \sin(\phi)\cos^2(\phi)\, d\phi \\
&= 16\pi\left(-\cos(\phi) + \frac{1}{3}\cos^3(\phi)\right)\Big|_0^{\pi} = 16\pi(1 - 1/3 - (-1+1/3)) = \frac{64}{3}\pi.
\end{aligned}$$

We can do the same thing for the smaller sphere oriented inwards, except we need to multiply our parametrization term by -1 so we get the orientation correct. We get

$$\begin{aligned}
\text{Flux} &= \int_0^{2\pi}\int_0^{\pi} (\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), 0)\cdot(-\sin\phi\cos\theta, -\sin\phi\sin\theta, -\cos\phi)\sin(\phi)\, d\phi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} -\sin^3\phi\cos^2\theta - \sin^3\phi\sin^2\theta\, d\phi\, d\theta \\
&= \int_0^{\pi}\int_0^{2\pi} -\sin^3\phi\, d\theta\, d\phi \\
&= \int_0^{\pi} -2\pi\sin^3(\phi)\, d\phi = -2\pi\int_0^{\pi} \sin(\phi) - \sin(\phi)\cos^2(\phi)\, d\phi \\
&= -2\pi\left(-\cos(\phi) + \frac{1}{3}\cos^3(\phi)\right)\Big|_0^{\pi} = -2\pi(1 - 1/3 - (-1+1/3)) = \frac{-8}{3}\pi.
\end{aligned}$$

So the total flux out of the sphere is $\frac{64}{3}\pi - \frac{8}{3}\pi = \frac{56}{3}\pi$.

(Note we didn't really need to work through the entire second integral; we know from the first part that $\int_0^{\pi}\sin^3(\phi)\, d\phi 4/3$ and that's enough to skip most of the second integral.)

Topic 14: Green's Theorem and Stokes's Theorem

(a) **Use Green's Theorem** to evaluate $\int_C (x\sin(y^2) - y^2)\, dx + (x^2 y\cos(y^2) + 3x)\, dy$ where C is the counterclockwise boundary of the trapezoid with vertices $(0,-2), (1,-1), (1,1), (0,2)$.

Solution:

We could parametrize the outside of the trapezoid, and then integrate that terrifying mess of a vector field, but that would be awful. Instead we use Green's Theorem to integrate the curl over the interior of the trapezoid. We compute the curl

$$\nabla\times\vec{F}(x,y) = \frac{\partial}{\partial x}(x^2 y\cos(y^2) + 3x) - \frac{\partial}{\partial y}(x\sin(y^2) - y^2) = 2xy\cos(y^2) + 3 - 2xy\cos(y^2) + 2y = 3 + 2y.$$

We see the trapezoid has $x \in [0,1]$, and y varies from $x-2$ to $2-x$. So by Green's Theorem we get

$$\begin{aligned}
\int_C \vec{F} \cdot d\vec{r} &= \iint_T 3 + 2y \, dA \\
&= \int_0^1 \int_{x-2}^{2-x} 3 + 2y \, dy \, dx \\
&= \int_0^1 3x + y^2 \Big|_{x-2}^{2-x} dy \\
&= \int_0^1 6 - 3x + (2-x)^2 - \left(3x - 6 + (x-2)^2\right) \, dx \\
&= \int_0^1 12 - 6x \, dx = 12x - 3x^2 \Big|_0^1 = 9.
\end{aligned}$$

(b) **Use Stokes's Theorem** to compute $\iint_S \nabla \times \vec{F} \cdot d\vec{S}$ where $\vec{F}(x,y,z) = (z-y, x, -x)$ and S is the hemisphere $x^2 + y^2 + z^2 = 4, z \geq 0$, oriented inwards towards the center of the hemisphere.

Solution:

Instead of parametrizing the spehre, we can just parametrize the boundary, which is $x^2 + y^2 = 4$ and parametrized by $(2\cos(t), 2\sin(t))$. However, we see this is the wrong orientation, so we instead use $\vec{r}(t) = (2\cos(t), -2\sin(t))$. (We could also take $(2\sin(t), 2\cos(t))$, or various other options.)

Thus we compute

$$\begin{aligned}
\iint_S \nabla \times \vec{F} \cdot d\vec{S} &= \int_C \vec{F} \cdot d\vec{R} \\
&= \int_0^{2\pi} (0 + 2\sin(t), 2\cos(t), -2\cos(t)) \cdot (-2\sin(t), -2\cos(t), 0) \, dt \\
&= \int_0^{2\pi} -4\sin^2(t) - 4\cos^2(t) \, dt \\
&= \int_0^{2\pi} -4 \, dt = -8\pi.
\end{aligned}$$