Math 2233 Fall 2021
Multivariable Calculus Mastery Quiz 2
Due Thursday, September 23

This week's mastery quiz has three topics. You may attempt all three topics. Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than about 20-30 minutes on this quiz.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in at class/recitation on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Topic 1: Planes
- Topic 2: Vector Operations
- Topic 3: Partial Derivatives and Linear Approximation

Name:

Recitation Section:

Topic 1: Planes

(a) Find an equation for the plane that passes through the points $(-1, 1, 8), (3, 0, -1)$, and $(2, 2, 3)$.

Solution:

$$\begin{aligned} z &= 8 + m(x+1) + n(y-1) \\ -1 &= 8 + m(3+1) + n(0-1) \\ 3 &= 8 + m(2+1) + n(2-1) \end{aligned}$$

which gives us the system of equations

$$\begin{aligned} -9 &= 4m - n \\ -5 &= 3m + n \end{aligned}$$

The first equation implies that $n = 4m + 9$ and thus the second gives us $-5 = 7m + 9$, which implies $m = -2$ and thus $n = 1$ Then the equation of our plane is

$$z = 8 - 2(x+1) + (y-1).$$

Alternatively, we can take the cross product of the vectors $4\vec{i} - \vec{j} - 9\vec{k}$ and $3\vec{i} + \vec{j} - 5\vec{k}$. This gives us

$$\begin{aligned} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 4 & -1 & -9 \\ 3 & 1 & -5 \end{vmatrix} &= 5\vec{i} - 27\vec{j} + 4\vec{k} - (-3\vec{k} - 9\vec{i} - 2\vec{0}) \\ &= 14\vec{i} - 7\vec{j} + 7\vec{k} \end{aligned}$$

and thus we get the equation

$$14(x+1) - 7(y-1) + 7(z-8) = 0.$$

(b) Find an equation for the plane perpendicular to $\vec{n} = \vec{i} + \vec{4}j - \vec{2}k$ that passes through the point $(5, -3, 0)$.

Solution:

$$(x-5) + 4(y-3) - 2(z-0) = 0.$$

(c) Find a vector perpendicular to the plane given by the equation

$$5(x-4) + 3(y+3) - 7(z-2) = 0$$

Solution:

A normal vector is $5\vec{i} + 3\vec{j} - 7\vec{k}$.

Topic 2: Vector Operations

(a) Find the orthogonal decomposition of $\vec{v} = 4\vec{i} + \vec{j} - \vec{k}$ with repsect to $\vec{u} = 2\vec{i} - \vec{j} + 3\vec{k}$.

Solution: First we compute the projection

$$\begin{aligned} \operatorname{Proj}_{\vec{u}} \vec{v} &= \frac{\vec{v} \cdot \vec{u}}{\vec{u} \cdot \vec{u}} \vec{u} \\ &= \frac{8-1-3}{4+1+9} \langle 2, -1, 3 \rangle \\ &= \frac{4}{14} \langle 2, -1, 3 \rangle = \frac{4}{7}\vec{i} - \frac{2}{7}\vec{j} + \frac{6}{7}\vec{k}. \end{aligned}$$

Now we still need the perpendicular component, but this is a straightforward subtraction:

$$\begin{aligned} \vec{v}_\perp &= \vec{v} - \operatorname{Proj}_{\vec{u}} \vec{v} \\ &= 4\vec{i} + \vec{j} - \vec{k} - \left(\frac{4}{7}\vec{i} - \frac{2}{7}\vec{j} + \frac{6}{7}\vec{k} \right) \\ &= \frac{24}{7}\vec{i} + \frac{9}{7}\vec{j} - \frac{13}{7}\vec{k}. \end{aligned}$$

(b) Find the area of the parallelogram with vertices $(0,0,0), (3,1,2), (2,4,3), (5,5,5)$.

Solution: This parallelogram is spanned by the vectors $3\vec{i} + \vec{j} + 2\vec{k}$ and $2\vec{i} + 4\vec{j} + 3\vec{k}$, so we compute

$$\begin{aligned} 3\vec{i} + \vec{j} + 2\vec{k} \times 2\vec{i} + 4\vec{j} + 3\vec{k} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 2 \\ 2 & 4 & 3 \end{vmatrix} \\ &= 3\vec{i} + 4\vec{j} + 12\vec{k} - 2\vec{k} - 8\vec{i} - 9\vec{j} \\ &= -5\vec{i} - 5\vec{j} + 10\vec{k}. \end{aligned}$$

Thus the area of the parallelogram is $\| -5\vec{i} - 5\vec{j} + 4\vec{k} \| = \sqrt{25 + 25 + 100} = \sqrt{150}$.

(c) Find $\cos\theta$ where θ is the angle between $\vec{u} = \vec{i} - 2\vec{j} + \vec{k}$ and $\vec{v} = 5\vec{i} + 2\vec{j} - 6\vec{k}$.

Solution:

$$\begin{aligned} \cos\theta &= \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|} \\ &= \frac{5-4-6}{\sqrt{6}\sqrt{65}} \\ &= \frac{-5}{\sqrt{330}}. \end{aligned}$$

Topic 3: Partial Derivatives and Linear Approximation

(a) Give an equation for the plane tangent to $f(x, y) = 3 + x^2y + \ln(x + xy)$ at the point $(1, 0)$.

Solution: We compute

$$\begin{aligned}
f_x(x,y) &= 2xy + \frac{1+y}{x+xy} \\
f_x(1,0) &= 0 + \frac{1}{1} = 1 \\
f_y(x,y) &= x^2 + \frac{x}{x+xy} \\
f_y(1,0) &= 1 + \frac{1}{1} = 2 \\
f(1,0) &= 3 + 0 + \ln(1+0) = 3 \\
z &= 3 + 1(x-1) + 2(y-0).
\end{aligned}$$

(b) Use a linear approximation to $g(x, y) = \sin(xy) + e^{xy^3}$ estimate $\sin(.1 \cdot 1.1) + e^{.1 \cdot 1.1^3}$.

Solution:

We take $g(x, y) = \sin(xy) + e^{xy^3}$ and compute

$$\begin{aligned}
g(0,1) &= \sin(0) + e^0 = 1 \\
g_x(x,y) &= y\cos(xy) + y^3e^{xy^3} \\
g_x(0,1) &= 1 \cdot 1 + 1 \cdot 1 = 2 \\
g_y(x,y) &= x\cos(xy) + 3xy^2e^{xy^3} \\
g_y(0,1) &= 0 + 0 = 0 \\
g(x,y) &\approx 1 + 2(x-0) + 0(y-1) = 1 + 2x \\
g(.1, 1.1) &\approx 1 + 2 \cdot .1 = 1.2.
\end{aligned}$$

(The exact answer is $1.25214\ldots$ so this isn't bad.)

(c) Let $h(x, y, z) = x^3yz^2 + \frac{xy}{y^2+z}$. Compute $h_x(x, y, z)$, $h_y(x, y, z)$, and $h_z(x, y, z)$.

Solution:

$$\begin{aligned}
h_x(x,y,z) &= 3x^2yz^2 + \frac{y}{y^2+z} \\
h_y(x,y,z) &= x^3z^2 + \frac{x(y^2+z) - 2y(xy)}{(y^2+z)^2} \\
h_z(x,y,z) &= 2x^3yz - \frac{xy}{(y^2+z)^2}.
\end{aligned}$$