Math 2233 Fall 2021
Multivariable Calculus Mastery Quiz 2
Due Thursday, September 23

This week's mastery quiz has four topics. **Submit no more than three**. If you already have a 2/2 on a topic, you should not submit it. (This may mean you only submit Topic 4, and that is perfectly fine.) This week will be the last week Topic 1 is on the quiz.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in at class/recitation on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Topic 1: Planes
- Topic 2: Vector Operations
- Topic 3: Partial Derivatives and Linear Approximation
- Topic 4: Gradient and Directional Derivatives

Name:

Recitation Section:

Topic 1: Planes

(a) Find an equation for the plane that passes through the points $(7, 3, -1), (4, 2, -2)$, and $(0, -1, 0)$.

Solution:

$$\begin{aligned} z &= -1 + m(x-7) + n(y-3) \\ -2 &= -1 + m(4-7) + n(2-3) \\ 0 &= -1 + m(0-7) + n(-1-3) \end{aligned}$$

which gives us the system of equations

$$\begin{aligned} -1 &= -3m - n \\ 1 &= -7m - 4n \end{aligned}$$

The first equation implies that $n = 1 - 3m$ and thus the second gives us $1 = -7m - 4 + 12m = 5m - 4$, which implies $m = 1$ and thus $n = -2$ Then the equation of our plane is

$$z = -1 + (x-7) - 2(y-3).$$

Alternatively, we can take the cross product of the vectors $3\vec{i} + \vec{j} + \vec{k}$ and $7\vec{i} + 4\vec{j} - \vec{k}$. This gives us

$$\begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 1 & 1 \\ 7 & 4 & -1 \end{vmatrix} = -\vec{i} + 7\vec{j} + 12\vec{k} - 7\vec{k} - 4\vec{i} + 3\vec{j} = -5\vec{i} + 10\vec{j} + 5\vec{k}$$

and thus we get the equation

$$-5(x-0) + 10(y+1) + 5(z-0) = 0.$$

(b) Find an equation for the plane perpendicular to $\vec{n} = 3\vec{i} - 2\vec{j} + 7\vec{k}$ that passes through the point $(2, 2, 2)$.

Solution:

$$3(x-2) - 2(y-2) + 7(z-2) = 0.$$

(c) Find a vector perpendicular to the plane given by the equation

$$-2(x-3) + 6(y+1) + 4(z+8) = 0$$

Solution:

A normal vector is $-2\vec{i} + 6\vec{j} + 4\vec{k}$.

Topic 2: Vector Operations

(a) Find the orthogonal decomposition of $\vec{v} = 2\vec{i} - 3\vec{j} + 5\vec{k}$ with repsect to $\vec{u} = \vec{i} + 3\vec{j} - 2\vec{k}$.

Solution: First we compute the projection

$$\begin{aligned}\text{Proj}_{\vec{u}}\,\vec{v} &= \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u}\\ &= \frac{-17}{14}(\vec{i} + 3\vec{j} - 2\vec{k})\\ &= \frac{-17}{14}\vec{i} - \frac{51}{14}\vec{j} + \frac{34}{14}\vec{k}\end{aligned}$$

Now we still need the perpendicular component, but this is a straightforward subtraction:

$$\begin{aligned}\vec{v}_{\perp} &= \vec{v} - \text{Proj}_{\vec{u}}\,\vec{v}\\ &= 2\vec{i} - 3\vec{j} + 5\vec{k} - \left(\frac{-17}{14}\vec{i} - \frac{51}{14}\vec{j} + \frac{34}{14}\vec{k}\right)\\ &= \frac{45}{14}\vec{i} + \frac{9}{14}\vec{j} + \frac{36}{14}\vec{k}.\end{aligned}$$

(b) Find the area of the **triangle** with vertices $(0,0,0), (5,3,1), (7,2,2)$.

Solution: This triangle is spanned by the vectors $5\vec{i} + 3\vec{j} + \vec{k}$ and $7\vec{i} + 2\vec{j} + 2\vec{k}$, so we compute

$$\begin{aligned}\left(5\vec{i} + 3\vec{j} + \vec{k}\right) \times \left(7\vec{i} + 2\vec{j} + 2\vec{k}\right) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & 3 & 1 \\ 7 & 2 & 2 \end{vmatrix}\\ &= 6\vec{i} + 7\vec{j} + 10\vec{k} - 21\vec{k} - 2\vec{i} - 10\vec{j}\\ &= 4\vec{i} - 3\vec{j} - 11\vec{k}.\end{aligned}$$

Thus the area of the triangle is $\frac{1}{2}\|4\vec{i} - 3\vec{j} - 11\vec{k}\| = \frac{1}{2}\sqrt{16 + 9 + 121} = \frac{1}{2}\sqrt{146} = \sqrt{73/2}$.

(c) Find $\cos\theta$ where θ is the angle between $\vec{u} = -2\vec{i} + 4\vec{j} + \vec{k}$ and $\vec{v} = 3\vec{i} + \vec{j} + 4\vec{k}$.

Solution:

$$\begin{aligned}\cos\theta &= \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|}\\ &= \frac{-6 + 4 + 4}{\sqrt{21}\sqrt{26}}\\ &= \frac{2}{\sqrt{546}}.\end{aligned}$$

Topic 3: Partial Derivatives and Linear Approximation

(a) Give an equation for the plane tangent to $f(x, y) = \cos(x^3y^2) - \frac{x}{y}$ at the point $(0, 3)$.

Solution: We compute

$$\begin{aligned}
f_x(x, y) &= -3x^2y^2 \sin(x^3y^2) - \frac{1}{y} \\
f_x(0, 3) &= 0 + \frac{1}{3} = 1/3 \\
f_y(x, y) &= -2y \sin(x^3y^2) - \frac{x}{y^2} \\
f_y(0, 3) &= 0 - 0 = 0 \\
f(0, 3) &= 1 - 0 = 1 \\
z &= 1 + 1/3(x - 0) + 0(y - 3) = 1 + x/3.
\end{aligned}$$

(b) Set $g(x, y) = x^2y^2 - \ln(2x - y)$. Use a linear approximation to estimate $g(1.1, .9)$.

Solution:

We compute

$$\begin{aligned}
g(1, 1) &= 1 - \ln(1) = 1 \\
g_x(x, y) &= 2xy^2 - \frac{2}{2x - y} \\
g_x(1, 1) &= 1 - \frac{2}{1} = -1 \\
g_y(x, y) &= 2x^2y + \frac{1}{2x - y} \\
g_y(1, 1) &= 1 + \frac{1}{1} = 2 \\
g(x, y) &\approx 1 - 1(x - 1) + 2(y - 1) \\
g(1.1, .9) &\approx 1 - 1(.1) + 2(-.1) = .7.
\end{aligned}$$

(The exact answer is $.717736\ldots$ so this isn't bad.)

(c) Let $h(x, y, z) = e^{xyz^2} - \frac{x+yz}{y^2-xz}$. Compute $\nabla h(x, y, z)$.

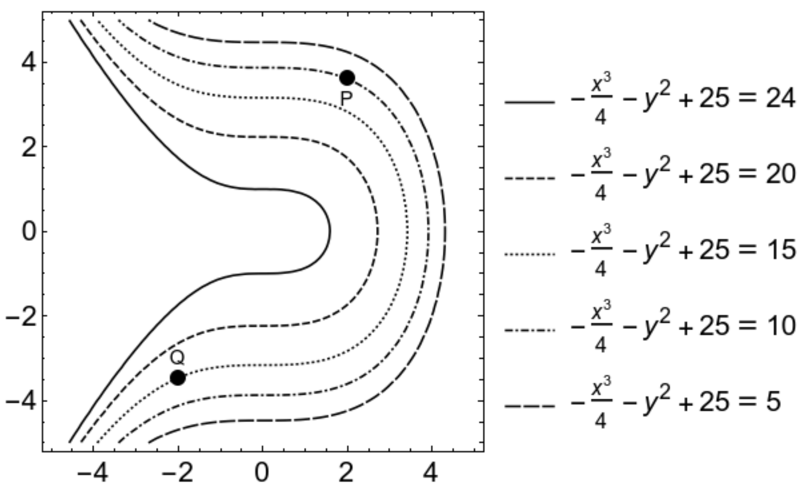
Solution:

$$\begin{aligned}
h_x(x, y, z) &= yz^2e^{xyz^2} - \frac{(y^2 - xz) + z(x + yz)}{(y^2 - xz)^2} \\
h_y(x, y, z) &= xz^2e^{xyz^2} - \frac{z(y^2 - xz) - 2y(x + yz)}{(y^2 - xz)^2} \\
h_z(x, y, z) &= 2xyze^{xyz^2} - \frac{y(y^2 - xz) + x(x + yz)}{(y^2 - xz)^2}.
\end{aligned}$$

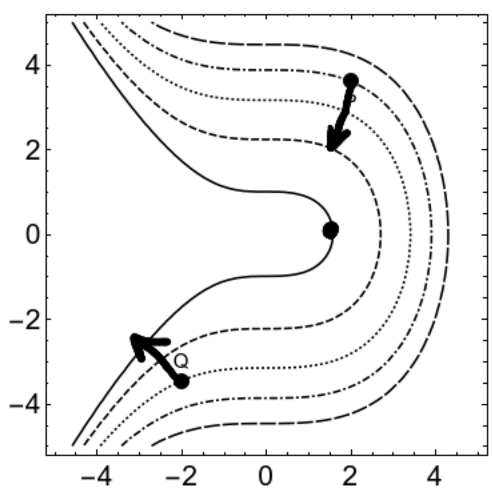
Topic 4: Gradients and Directional Derivatives

(a) Below is a contour plot of the function $h(x, y)$.

- Sketch the gradient vector at P.
- Sketch the gradient vector at Q.
- Label a point on the diagram where $\frac{\partial h}{\partial y} = 0$.



Solution:



(b) Let $f(x, y, z) = x^2y - yz^3$. Find the directional derivative in the direction $\vec{i} + 2\vec{j} - \vec{k}$ at the point $(1, 2, 1)$.

Solution: We compute

$$\begin{aligned}
\nabla f(x,y,z) &= \langle 2xy, x^2 - z^3, -3yz^2\rangle \\
\nabla f(1,2,1) &= \langle 4, 0, -6\rangle \\
\vec{u} &= \frac{1}{\sqrt{6}}\vec{i} + \frac{2}{\sqrt{6}}\vec{j} - \frac{1}{\sqrt{6}}\vec{k} \\
f_{\vec{u}}(1,2,1) &= \nabla f(1,2,1)\cdot \vec{u} \\
&= \frac{4}{\sqrt{6}} + \frac{6}{\sqrt{6}} = \frac{10}{\sqrt{6}}.
\end{aligned}$$

(c) Find all three second partial derivatives of $g(x,y) = \sqrt{x^2+y}$.

Solution:

$$\begin{aligned}
g_x(x,y) &= \frac{x}{\sqrt{x^2+y}} \\
g_y(x,y) &= \frac{1}{2\sqrt{x^2+y}} \\
g_{xx}(x,y) &= \frac{\sqrt{x^2+y} - \frac{x^2}{\sqrt{x^2+y}}}{(x^2+y)} \\
g_{xy}(x,y) &= \frac{-2x}{4(x^2+y)^{3/2}} \\
g_{yy}(x,y) &= \frac{-1}{4(x^2+xy)^{3/2}}.
\end{aligned}$$