Math 2233 Fall 2021
Multivariable Calculus Mastery Quiz 4
Due Thursday, October 14

This week's mastery quiz has five topics. **Submit no more than three**. If you already have a 2/2 on a topic, you should not submit it. Please **check Blackboard for updated scores**, since your midterm performance can impact your mastery score. You may only need to submit topic 6; please do submit topic 6 regardless of your other choices. This week will be the last week Topics 2 and 3 are on the quiz.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in at class/recitation on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Topic 2: Vector Operations
- Topic 3: Partial Derivatives and Linear Approximation
- Topic 4: Gradient and Directional Derivatives
- Topic 5: Multivariable Optimization
- Topic 6: Constrained Optimization

Name:

Recitation Section:

Topic 2: Vector Operations

(a) Find the orthogonal decomposition of $\vec{v} = 6\vec{i} + 2\vec{j} - 3\vec{k}$ with repsect to $\vec{u} = 2\vec{i} + \vec{j} + 2\vec{k}$.

Solution: First we compute the projection

$$\begin{aligned} \text{Proj}_{\vec{u}}\,\vec{v} &= \frac{\vec{v}\cdot\vec{u}}{\vec{u}\cdot\vec{u}}\vec{u} \\ &= \frac{8}{9}(2\vec{i} + \vec{j} + 2\vec{k}) \\ &= \frac{16}{9}\vec{i} + \frac{8}{9}\vec{j} + \frac{16}{9}\vec{k} \end{aligned}$$

Now we still need the perpendicular component, but this is a straightforward subtraction:

$$\begin{aligned} \vec{v}_\perp &= \vec{v} - \text{Proj}_{\vec{u}}\,\vec{v} \\ &= 6\vec{i} + 2\vec{j} - 3\vec{k} - \left(\frac{16}{9}\vec{i} + \frac{8}{9}\vec{j} + \frac{16}{9}\vec{k}\right) \\ &= \frac{38}{9}\vec{i} + \frac{10}{9}\vec{j} - \frac{43}{9}\vec{k}. \end{aligned}$$

(b) Find the area of the parallelogram with vertices $(0,0,0), (2,2,2), (1,5,5), (3,7,7)$.

Solution: This parallelogram is spanned by the vectors $2\vec{i} + 2\vec{j} + 2\vec{k}$ and $\vec{i} + 5\vec{j} + 5\vec{k}$, so we compute

$$\begin{aligned} \left(2\vec{i} + 2\vec{j} + 2\vec{k}\right) \times \left(1\vec{i} + 5\vec{j} + 5\vec{k}\right) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & 2 \\ 1 & 5 & 5 \end{vmatrix} \\ &= 10\vec{i} + 2\vec{j} + 10\vec{k} - 2\vec{k} - 10\vec{i} - 10\vec{j} \\ &= -8\vec{j} + 8\vec{k}. \end{aligned}$$

Thus the area of the parallelogram is $\| -8\vec{j} + 8\vec{k}\| = \sqrt{64+64} = \sqrt{128} = 8\sqrt{2}$.

(c) Find $\cos\theta$ where θ is the angle between $\vec{u} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{v} = \vec{i} - \vec{j} + \vec{k}$.

Solution:

$$\begin{aligned} \cos\theta &= \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\|\vec{v}\|} \\ &= \frac{1-2+3}{\sqrt{14}\sqrt{3}} \\ &= \frac{2}{\sqrt{42}}. \end{aligned}$$

Topic 3: Partial Derivatives and Linear Approximation

(a) Give an equation for the plane tangent to $f(x,y) = x\sin(xy)$ at the point $(1,\pi)$.

Solution: We compute

$$\begin{aligned} f_x(x,y) &= \sin(xy) + xy\cos(xy) \\ f_x(1,\pi) &= -\pi \\ f_y(x,y) &= x^2\cos(xy) \\ f_y(1,\pi) &= -1 \\ f(1,\pi) &= 0 \\ z &= 0 - \pi(x-1) - (y-\pi). \end{aligned}$$

(b) Set $g(x,y) = ye^{2x+y}$. Use a linear approximation to estimate $g(-.9, 2.1)$.

Solution:

We compute

$$\begin{aligned} g(-1,2) &= 2 \\ g_x(x,y) &= 2ye^{2x+y} \\ g_x(-1,2) &= 4 \\ g_y(x,y) &= e^{2x+y} + ye^{2x+y} \\ g_y(-1,2) &= 3 \\ g(x,y) &\approx 2 + 4(x+1) + 3(y-2) \\ g(-.9,2.1) &\approx 2 + .4 + .3 = 2.7 \end{aligned}$$

(The exact answer is $2.8347\ldots$ so this isn't great but isn't too bad.)

(c) Let $h(x,y,z) = \tan(x^2y + y^2z) - xz$. Compute $\nabla h(x,y,z)$.

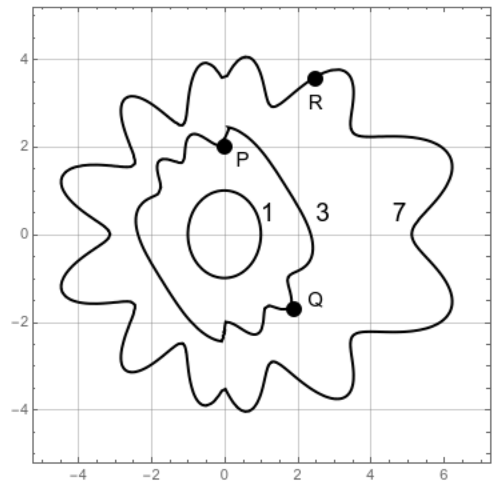
Solution:

$$\begin{aligned} h_x(x,y,z) &= \sec^2(x^2y + y^2z)\cdot 2xy - z \\ h_y(x,y,z) &= \sec^2(x^2y + y^2z)(x^2 + 2yz) \\ h_z(x,y,z) &= \sec^2(x^2y + y^2z)y^2 - x. \end{aligned}$$

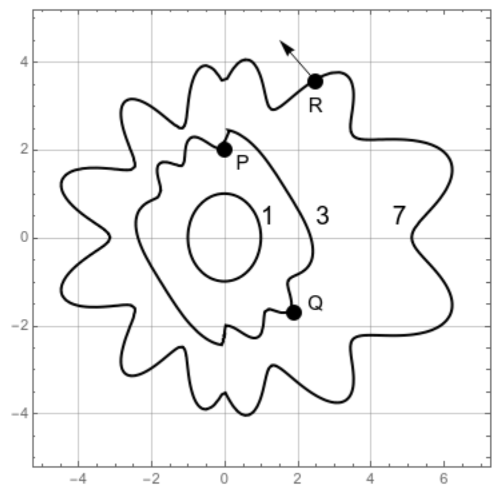
Topic 4: Gradients and Directional Derivatives

(a) Below is a contour plot of the function $h(x,y)$.

- Sketch the gradient vector at R.
- Estimate $\frac{\partial h}{\partial y}$ at the point P. Explain your reasoning in a sentence or so.



Solution:



At the point p, we see that going up by two units (one gridline, from 2 to 4) increases the output by about 4. Going down by one unit decreases the output by about 2. Consequently, the derivative should be about 2.

(b) Let $f(x, y, z) = \ln(xy + z)$. Find the directional derivative in the direction $-\vec{i} - \vec{j} + \vec{k}$ at the point $(0, 3, 1)$.

Solution: We compute

$$\begin{aligned}
\nabla f(x, y, z) &= \langle \frac{y}{xy+z}, \frac{x}{xy+z}, \frac{1}{xy+z} \rangle \\
\nabla f(0, 3, 1) &= \langle 3, 0, 1 \rangle \\
\vec{u} &= \frac{-1}{\sqrt{3}}\vec{i} - \frac{1}{\sqrt{3}}\vec{j} + \frac{1}{\sqrt{3}}\vec{k} \\
f_{\vec{u}}(0, 3, 1) &= \nabla f(0, 3, 1) \cdot \vec{u} \\
&= -\frac{3}{\sqrt{3}} + \frac{1}{\sqrt{3}} = \frac{-2}{\sqrt{3}}.
\end{aligned}$$

(c) Find all three second partial derivatives of $g(x, y) = x^2y + xy^3$.

Solution:

$$\begin{aligned} g_x(x,y) &= 2xy + y^3 \\ g_y(x,y) &= x^2 + 3xy^2 \\ g_{xx}(x,y) &= 2y \\ g_{xy}(x,y) &= 2x + 3y^2 \\ g_{yy}(x,y) &= 6xy. \end{aligned}$$

Topic 5: Multivariable Optimization

(a) Find and classify the critical points of $f(x,y) = (3x + 4x^3)(y^2 + 2y)$.

Solution: We have

$$\begin{aligned} f_x(x,y) &= (3 + 12x^2)(y^2 + 2y) \\ f_y(x,y) &= (3x + 4x^3)(2y + 2). \end{aligned}$$

The first equation tells us that we must have $y^2 + 2y = 0$ and thus $y = 0$ or $y = -2$. In either case, the second equation gives us that $3x + 4x^3 = 0$ and thus $x = 0$. (There are no other roots, since $3 + 4x^2 \neq 0$ for any real x. We don't want to think about complex numbers here.)

Thus our critical points are $(0,0)$ and $(0,-2)$.

We have

$$\begin{aligned} f_{xx}(x,y) &= 24x(y^2 + 2y) & f_{xx}(0,0) &= 0 & f_{xx}(0,-2) &= 0 \\ f_{xy}(x,y) &= (3 + 12x^2)(2y + 2) & f_{xy}(0,0) &= 6 & f_{xy}(0,-2) &= -6 \\ f_{yy}(x,y) &= 6x + 8x^3 & f_{yy}(0,0) &= 0 & f_{yy}(0,-2) &= 0. \end{aligned}$$

Then for $(0,0)$ we have $D = 0 \cdot 0 - 6^2 = -36 < 0$, so we have a saddle point.

For $(0,-2)$ we have $D = 0 \cdot 0 - (-6)^2 = -36 < 0$, and so this is also a saddle point.

(b) Find (but don't classify) the critical points of $g(x,y,z) = x^2y - xz^2$.

Solution: We have

$$\begin{aligned} g_x(x,y,z) &= 2xy - z^2 \quad g_y(x,y,z) &= x^2 \\ g_z(x,y,z) &= -2xz \end{aligned}$$

The second equation says that $x = 0$. Then the first says that $z = 0$ and the third is totally redundant, so we have a critical point whenever $x = z = 0$.

(Important note: y is totally unconstrained here! So there are infinitely many critical points.)

Topic 6: Constrained Optimization

(a) Find the maximum and minimum values of $f(x,y) = xy$ subject to the constraint that $x^2 + 4y^2 \leq 1$.

Solution:

We need to think about interior critical points, and also critical points on the boundary.

We have $\nabla f = \langle y, x \rangle$ which is zero only when $x, y = 0$. This gives us one critical point at $(0,0)$, and $f(0,0) = 0$.

Now we consider the boundary. We have

$$\begin{aligned} y &= \lambda 2x \\ x &= \lambda 8y. \end{aligned}$$

This gives us $x = \lambda^2 \cdot 16x$ and so $\lambda = \pm 1/4$. (Or $x = 0$, but then we have $y = 0$ and that contradicts the constraint equation.)

Then we have $x = \pm 2y$ and thus our equation is $4y^2 + 4y^2 = 1$, so $y = \pm\sqrt{1/8}$. Then $x = \pm\sqrt{1/2}$, and all four combinations are possible. We calculate

$$\begin{aligned} f(1/\sqrt{2}, 1/\sqrt{8} &= \frac{1}{\sqrt{16}} = 1/4 \\ f(1/\sqrt{2}, -1/\sqrt{8}) &= -1/4 \\ f(-1/\sqrt{2}, 1/\sqrt{8} &= -1/4 \\ f(-1/\sqrt{2}, -1/\sqrt{8}) &= 1/4. \end{aligned}$$

Thus the absolute maximum of f on this region is $1/4$, and the absolute minimum is $-1/4$.

(b) Find the maximum and minimum values of $g(x,y,z) = y^2 - 10z$ subject to the constraint $x^2 + y^2 + z^2 = 36$.

Solution:

We get the three equations

$$\begin{aligned} 0 &= \lambda \cdot 2x \\ 2y &= \lambda 2y \\ -10 &= \lambda 2z \end{aligned}$$

The first equation tells us that $x = 0$ *or* that $\lambda = 0$. Note that either of these things can happen! We could analyze both these cases, but I want to see if some equation is more useful than the other.

The second equation, similarly, tells us that $\lambda = 1$ *or* that $y = 0$. Again, we can analyze either case.

The third equation tells us that $z = -5/\lambda$. In particular, we see that neither z nor λ can be zero.

So now we can go back to the first equation; we know $\lambda \neq 0$ and thus $x = 0$. The second equation genuinely gives us two possibilities.

If $\lambda = 1$, then $z = -5$; from the constraint equation we have $y^2+25 = 36$ so $y = \pm\sqrt{11}$. Our two critical poitns are $(0, \sqrt{11}, -5)$ and $(0, -\sqrt{11}, -5)$. We compute

$$g(0, \sqrt{11}, -5) = 11 + 50 = 61$$
$$g(0, -\sqrt{11}, -5) = 11 + 50 = 61.$$

(If you hadn't noticed the other two critical points, you should realize now that something has gone wrong: the function isn't constant but you've only found one value.)

If $y = 0$ then the constraint equation gives us $z^2 = 36$ so $z = \pm 6$. Our critical points here are $(0, 0, 6)$ and $(0, 0, -6)$ and so we compute

$$g(0, 0, 6) = -60$$
$$g(0, 0, -6) = 60.$$

Thus the globam maximum is 61, achieved at $(0, \sqrt{11}, -5)$ and $(0, -\sqrt{11}, -5)$; the global minimum is -60, achieved at $(0, 0, 6)$.