Math 2233 Fall 2021
Multivariable Calculus Mastery Quiz 5
Due Thursday, October 14

This week's mastery quiz has four topics. **Submit no more than three**. If you already have a 2/2 on a topic, you should not submit it. Please **check Blackboard for updated scores**, since your midterm performance can impact your mastery score. You may only need to submit topic 6; please do submit topic 6 regardless of your other choices. This week will be the last week Topics 2 and 3 are on the quiz.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in at class/recitation on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Topic 4: Gradient and Directional Derivatives
- Topic 5: Multivariable Optimization
- Topic 6: Constrained Optimization
- Topic 7: Multivariable Integrals

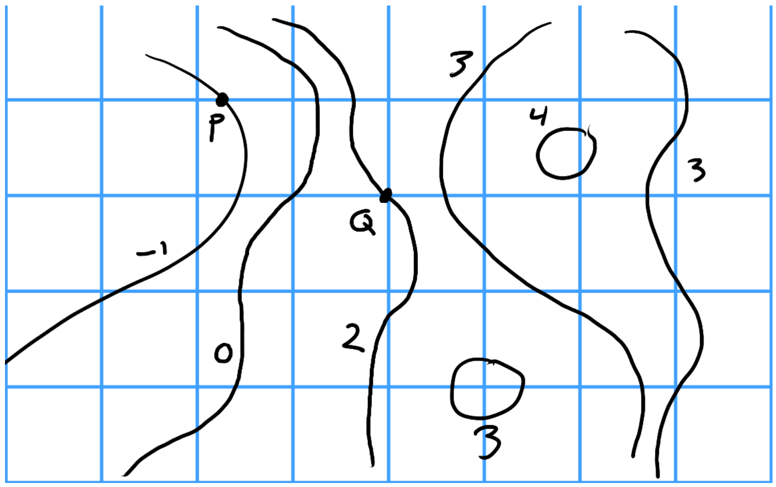
Name:

Recitation Section:

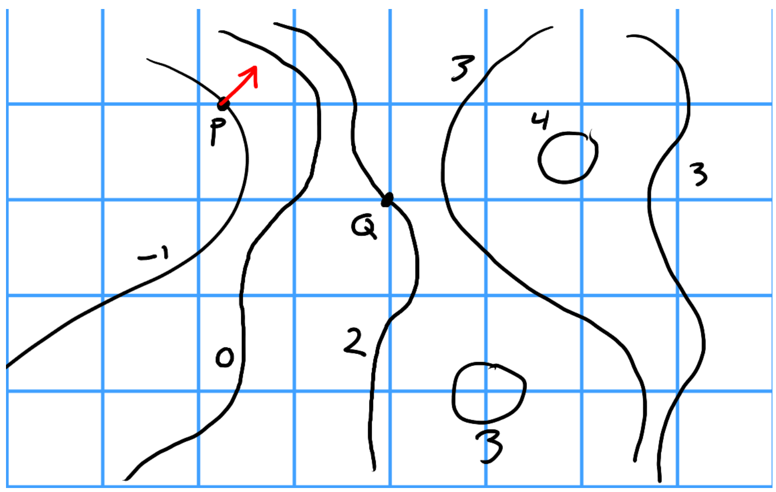
Topic 4: Gradients and Directional Derivatives

(a) Below is a contour plot of the function $f(x, y)$.

- Sketch the gradient vector at P.
- Estimate $\frac{\partial h}{\partial x}$ at the point Q. Explain your reasoning in a sentence or so.



Solution:



It looks like $\frac{partial f}{\partial x}$ is about 2. If you move one unit to the left, your output drops by about 2. If you move about half a unit to the right, your value increases by 1.

(b) Let $f(x, y, z) = e^{x\sqrt{y}}/z$. Find the directional derivative in the direction $\vec{i} + 3\vec{j} - 2\vec{k}$ at the point $(0, 1, 1)$.

Solution: We compute

$$\begin{aligned}
\nabla f(x,y,z) &= \langle \sqrt{y}e^{x\sqrt{y}}/z, \frac{x}{2\sqrt{y}z}e^{x\sqrt{y}}, -e^{x\sqrt{y}}/z^2\rangle \\
\nabla f(0,1,1) &= \langle 1,0,-1\rangle \\
\vec{u} &= \frac{1}{\sqrt{14}}\vec{i} + \frac{3}{\sqrt{14}}j - \frac{2}{\sqrt{2}}\vec{k} \\
f_{\vec{u}}(0,1,1) &= \nabla f(0,1,1)\cdot \vec{u} \\
&= \langle 1,0,-1\rangle \cdot \langle \frac{1}{\sqrt{14}}, \frac{3}{\sqrt{14}}, -\frac{2}{\sqrt{14}}\rangle \qquad = \frac{1}{\sqrt{14}} + \frac{2}{\sqrt{14}} = \frac{3}{\sqrt{14}}
\end{aligned}$$

(c) Find all second partial derivatives of $g(x,y) = \frac{x^2-3}{y+1}$.

Solution:

$$\begin{aligned}
g_x(x,y) &= \frac{2x}{y+1} \\
g_y(x,y) &= \frac{3-x^2}{(y+1)^2} \\
g_{xx}(x,y) &= \frac{2}{y+1} \\
g_{xy}(x,y) &= -2x(y+1)^2 \\
g_{yy}(x,y) &= \frac{2x^2-6}{(y+1)^3}.
\end{aligned}$$

Topic 5: Multivariable Optimization

(a) Find and classify the critical points of $f(x,y) = x^2 + xy^2 + y^2$.

Solution: We have

$$\begin{aligned}
f_x(x,y) &= 2x + y^2 \\
f_y(x,y) &= 2xy + 2y
\end{aligned}$$

The second equation tells us that either $y = 0$ or $x = -1$. If $y = 0$ then the first equation gives $x = 0$. If $x = -1$ then the first equation gives $y^2 = 2$ so $y = \pm\sqrt{2}$. Thus our critical points are $(0,0), (-1,\sqrt{2}), (-1,-\sqrt{2})$.

We have

$$\begin{array}{llll}
f_{xx}(x,y) = 2 & f_{xx}(0,0) = 2 & f_{xx}(-1,\sqrt{2}) = 2 & f_{xx}(-1,-\sqrt{2}) = 2 \\
f_{xy}(x,y) = 2y & f_{xy}(0,0) = 0 & f_{xy}(-1,\sqrt{2}) = 2\sqrt{2} & f_{xy}(-1,-\sqrt{2}) = -2\sqrt{2} \\
f_{yy}(x,y) = 2 & f_{yy}(0,0) = 2 & f_{yy}(-1,\sqrt{2}) = 2 & f_{yy}(-1,-\sqrt{2}) = 2
\end{array}$$

Then for $(0,0)$ we have $D = 2\cdot 2 - 0^2 = 4 > 0$, and since $f_{xx}(0,0) = 2 > 0$ this is a local minimum.

For $(-1, \sqrt{2})$ we have $D = 2 \cdot 2 - (2\sqrt{2})^2 = -4 < 0$, so this is a saddle point.

For $(-1, -\sqrt{2})$ we have $D = 2 \cdot 2 - (-2\sqrt{2})^2 = -4 < 0$, so this is a saddle point.

(b) Find (but don't classify) the critical points of $g(x, y, z) = x^2 + y^2 + 3z^2 - 2x - 8y - z^3 + 5$.

Solution: We have

$$\begin{aligned} g_x(x, y, z) &= 2x - 2 \\ g_y(x, y, z) &= 2y - 8 \\ g_z(x, y, z) &= 6z - 3z^2 \end{aligned}$$

The first equation says $x = 1$, and the second says $y = 4$. Then the third gives us $3z(2 - z) = 0$, and so $z = 0$ or $z = 2$. So there are two critical points: $(1, 4, 0)$ and $(1, 4, 2)$.

Topic 6: Constrained Optimization

Find the maximum and minimum values of $f(x, y, z) = x + y^2 + 3z$ given that $4x^2 + 8y^2 + 36z^2 \leq 36$.

Solution:

We first look for interior critical points. We have $\nabla f(x, y, z) = \langle 1, 2y, 3 \rangle$. This is never zero, so we have no interior critical points. We now consider the boundary.

We have

$$\begin{aligned} 1 &= \lambda 8x \\ 2y &= \lambda 16y \\ 3 &= \lambda 72z. \end{aligned}$$

The second equation tells us that $y = 0$ or $\lambda = 1/8$.

If $\lambda = 1/8$, then the first equation tells us that $x = 1$ and $z = 1/3$. Plugging this into the constraint equation gives us

$$\begin{aligned} 4 + 8y^2 + 4 &= 36 \\ 8y^2 &= 28 \\ y &= \pm\sqrt{7/2}. \end{aligned}$$

This gives us two critical points: $(1, \sqrt{7/2}, 1/3)$ and $(1, -\sqrt{7/2}, 1/3)$. We compute

$$\begin{aligned} f(1, \sqrt{7/2}, 1/3) &= 1 + 7/2 + 1 = 11/2 \\ f(1, -\sqrt{7/2}, 1/3) &= 1 + 7/2 + 1 = 11/2. \end{aligned}$$

If $y = 0$, then we have

$$\begin{aligned} \lambda &= \frac{1}{8x} \\ \lambda &= \frac{1}{24z} \\ 24z &= 8x \\ x &= 3z \end{aligned}$$

and thus

$$\begin{aligned} 36z^2 + 36z^2 &= 36 \\ z^2 &= 1/2 \end{aligned}$$

and thus $z = \pm 1/\sqrt{2}$ and $x = \pm 3/\sqrt{2}$. Then we have two critical points, and we compute

$$\begin{aligned} f(3/\sqrt{2}, 0, 1/\sqrt{2}) &= 3/\sqrt{2} + 3/\sqrt{2} = 6/\sqrt{2} \\ f(-3/\sqrt{2}, 0, -1/\sqrt{2}) &= -3/\sqrt{2} - 3/\sqrt{2} = -6/\sqrt{2} \end{aligned}$$

So the maximum value is $11/2$ and the minimum value is $-6\sqrt{2}$.

Topic 7: Multivariable integrals

(a) Sketch the region of integration and compute $\int_{-1}^{0} \int_{-\sqrt{y+1}}^{\sqrt{y+1}} y^2 \, dx \, dy$. (Do not use a calculator!)

Solution:

This is much easier if we change the order of integration. We see that x varies from -1 to 1, and then the equation of the curve is $x^2 = y + 1$ and thus $y = x^2 - 1$. So we have

$$\begin{aligned} \int_{-1}^{1} \int_{x^2-1}^{0} y^2 \, dy \, dx &= \int_{-1}^{1} \frac{y^3}{3} \Big|_{x^2-1}^{0} dx \\ &= \int_{-1}^{1} -\frac{(x^2-1)^3}{3} \, dx = \frac{1}{3} \int_{-1}^{1} 1 - 3x^2 + 3x^4 - x^6 \, dx \\ &= \frac{1}{3} \left(x - x^3 + \frac{3}{5}x^5 - \frac{1}{7}x^7 \right) \Big|_{-1}^{1} \\ &= \frac{1}{3} \left(1 - 1 + 3/5 - 1/7 - (-1 + 1 - 3/5 + 1/7)\right) = 6/15 - 2/21 = \frac{32}{105}. \end{aligned}$$

(b) Integrate the function $f(x, y, z) = xyz$ over the region bounded by $x = 0, x = 4, y = 0, z = 0$, and $y = 4 - z^2$.

Solution:

x goes from 0 to 4. Then z goes from 0 to 2 and y goes from 0 to $4 - z^2$, and we have

$$\begin{aligned}
\int_0^2 \int_0^{4-z^2} \int_0^4 xyz\,dx\,dy\,dz &= \int_0^2 \int_0^{4-z^2} \left.\frac{x^2yz}{2}\right|_0^4 dy\,dz \\
&= \int_0^2 \int_0^{4-z^2} 8yz\,dy\,dz \\
&= \int_0^2 \left.4y^2z\right|_0^{4-z^2} dz \\
&= \int_0^2 4z(4-z^2)^2\,dz = 4\int_0^2 z^5 - 8z^3 + 16z\,dz \\
&= 4\left(\frac{z^6}{6} - 2z^4 + 8z^2\right)\Bigg|_0^2 \\
&= 4\left(\frac{32}{3} - 32 + 32\right) = 128/3.
\end{aligned}$$