Math 2233 Fall 2021
Multivariable Calculus Mastery Quiz 6
Due Thursday, October 28

This week's mastery quiz has five topics. **Submit no more than four**. If you already have a 2/2 on a topic, you should not submit it. Please **check Blackboard for updated scores**, since your midterm performance can impact your mastery score. You may only need to submit topics 8 and 9. This week will be the last week Topics 5 and 6 are on the quiz.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in at class/recitation on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Topic 5: Multivariable Optimization
- Topic 6: Constrained Optimization
- Topic 7: Multivariable Integrals
- Topic 8: Integrals in Other Coordinates
- Topic 9: Calculus of Curves

Name:

Recitation Section:

Topic 5: Multivariable Optimization

(a) Find and classify the critical points of $f(x, y) = 2x^3 - 6xy + y^2$.

Solution: We have

$$f_x(x, y) = 6x^2 - 6y$$
$$f_y(x, y) = -6x + 2y$$

The second equation tells us $y = 3x$. Substituting that into the first equation gives $6x^2 - 18x = 0$ and thus either $x = 0$ or $x = 3$. So our two critical points are $(0, 0)$ and $(3, 9)$.

We have

$$\begin{array}{lll} f_{xx}(x, y) = 12x & f_{xx}(0, 0) = 0 & f_{xx}(3, 9) = 36 \\ f_{xy}(x, y) = -6 & f_{xy}(0, 0) = -6 & f_{xy}(3, 9) = -6 \\ f_{yy}(x, y) = 2 & f_{yy}(0, 0) = 2 & f_{yy}(3, 9) = 2 \end{array}$$

Then for $(0, 0)$ we have $D = 0 \cdot 2 - (-6)^2 = -36 < 0$, so this is a saddle point.

For $(3, 9)$ we have $D = 36 \cdot 2 - (-6)^2 = 36 > 0$, and $f_{xx} = 36 > 0$, so this is a local minimum.

(b) Find (but don't classify) the critical points of $g(x, y, z) = x^3 + y^3 - 3x^2 - y^2 - z^2 + 2z - 1$.

Solution: We have

$$g_x(x, y, z) = 3x^2 - 6x$$
$$g_y(x, y, z) = 3y^2 - 2y$$
$$g_z(x, y, z) = -2z + 2$$

The first equation gives $x = 0$ or $x = 2$; the second equation gives $y = 0$ or $y = 2/3$; and the third equation gives $z = 1$. So the critical points are

$$(0, 0, 1), (2, 0, 1), (0, 2/3, 1), (2, 2/3, 1).$$

Topic 6: Constrained Optimization

Use the method of Lagrange multipliers to find the point on the circle $x^2 + y^2 = 40$ closest to the point $(1, 3)$.

Solution:

Our constraint function is $g(x, y) = x^2 + y^2 = 40$. Our objective function is the distance $\sqrt{(x-1)^2 + (y-3)^2}$, but it's okay to just minimize $f(x, y) = (x-1)^2 + (y-3)^2$. Then we have

$$2(x - 1) = \lambda 2x$$
$$2(y - 3) = \lambda 2y.$$

Rearranging gives

$$\begin{aligned}(1-\lambda)x &= 1\\ (1-\lambda)y &= 3\\ 1/x &= 3/y\\ y &= 3x.\end{aligned}$$

Substituting that into our constraint equation gives

$$\begin{aligned}x^2 + 9x^2 &= 40\\ x^2 &= 4\\ x &= \pm 2\end{aligned}$$

and thus our critical points are $(2,6)$ and $(-2,-6)$ which we check are indeed on the circle.

Then we compute $f(2,6) = 1^2 + 3^2 = 10$, so this point is at distance $\sqrt{10}$ from $(1,3)$. And we have $f(-2,-6) = 3^2 + 9^2 = 90$ so this point is at distance $\sqrt{90}$ from $(1,3)$. Thus the point closest to $(1,3)$ is $(2,6)$—which maybe becomes obvious in retrospect if you look at a picture.

Topic 7: Multivariable integrals

(a) Sketch the region of integration and compute $\iint_R xy^2\,dx\,dy$, where R is the region in the first quadrant bounded by the curves $y = x^2$ and $x = y^2$. (Do not use a calculator!)

Solution:

These curves intersect at $(0,0)$ and $(1,1)$. So we can take x going from 0 to 1, and then y goes from x^2 to $\sqrt{x}$. (Coutnerintuitively, $x^2 < \sqrt{x}$ in this region.)

So we compute

$$\begin{aligned}\iint_R xy^2\,dx &= \int_0^1 \int_{x^2}^{\sqrt{x}} xy^2\,dy\,dx\\ &= \int_0^1 xy^3/3\Big|x^{2\sqrt{x}}\,dx\\ &= \frac{1}{3}\int_0^1 x^{5/2} - x^7\,dx\\ &= \frac{1}{3}\left(\frac{2}{7}x^{7/2} - \frac{1}{8}x^8\right)\Big|\,0^1\\ &= \frac{1}{3}\left(2/7 - 1/8\right) = \frac{3}{56}.\end{aligned}$$

We could have set it up in the other order. Then we would get

$$\begin{aligned}\iint_R xy^2\,dx &= \int_0^1 \int_{y^2}^{\sqrt{y}} xy^2\,dx\,dy \\ &= \int_0^1 x^2y^2/2\Big|_{y^2}^{\sqrt{y}}\,dy \\ &= \frac{1}{2}\int_0^1 y^3 - y^6\,dy \\ &= \frac{1}{2}\left(y^4/4 - y^7/7\right)\Big|_0^1 \\ &= \frac{1}{2}(1/4 - 1/7) = \frac{3}{56}.\end{aligned}$$

I think the second way looks easier, but it's up to you! (When I worked this out, I got to the point where I was cubing $\sqrt{x}$ and realized if I did it in the other order things would work out better.)

(b) Find the mass of the tetrahedron bounded by the planes $x = 0, y = 0, z = 0$, and $x + 2y + 3z = 6$ if the density is given by $\delta(x, y, z) = z$.

Solution:

z goes from 0 to 2. Then y goes from 0 to $6 - 3z/2$ and x goes from 0 to $6 - 2y - 3z$, and we have

$$\begin{aligned}\int_0^2 \int_0^{6-3z/2} \int_0^{3-2y-3z} z\,dx\,dy\,dz &= \int_0^2 \int_0^{3-3z/2} z\Big|_0^{6-2y-3z}\,dy\,dz \\ &= \int_0^2 \int_0^{3-3z/2} 6z - 2yz - 3z^2\,dy\,dz \\ &= \int_0^2 6yz - y^2z - 3yz^2\Big|_0^{3-3z/2}\,dz \\ &= \int_0^2 18z - 9z^2 - (3-3z/2)^2z - 9z^2 + 9z^3/2\,dz \\ &= \int_0^2 18z - 18z^2 + 9z^3/2 - 9z + 9z^2 - 9z^3/4\,dz \\ &= \int_0^2 9z - 9z^2 + 9z^3/4\,dz \\ &= 9z^2/2 - 3z^3 + 9z^4/16\Big|_0^2 = 18 - 24 + 9 = 3.\end{aligned}$$

Topic 8: Integrals in Other Coordinate Systems

(a) Sketch the region of integration and compute $\iint_R y\sqrt{x^2+y^2}\,dA$ where R is the region given by $x^2 + y^2 \leq 4$ and $0 \leq y \leq x$.

Solution:

We change to polar coordinates because this is a wedge of a circle. We have $0 \leq r \leq 2$ (be careful here, $x^2 + y^2$ is the radius squared!) and $0 \leq theta \leq \pi/4$. We know that $\sqrt{x^2 + y^2} = r$ and $y = r \sin\theta$, so we get

$$\begin{aligned}
\iint_R y\sqrt{x^2+y^2}\, dA &= \int_0^2 \int_0^{\pi/4} r^2 \sin(\theta) \cdot r \, d\theta \, dr \\
&= \int_0^2 -r^3 \cos(\theta)\Big|_0^{\pi/4} dr \\
&= \int_0^2 -r^3(\sqrt{2}/2 - 1)\, dr \\
&= (1 - \sqrt{2}/2) r^4/4 \Big|_0^3 = 4 - 4\sqrt{2}/2.
\end{aligned}$$

(b) Find the volume of the solid bounded by the paraboloid $z = x^2 + y^2$ and the sphere $x^2 + y^2 + z^2 = 6$ (above the plane z=0).

Solution:

We see that the two surfaces intersect at $z + z^2 = 6$, which has the solutions $z = 2$ and $z = -3$. But we can't have $z = -3$ when $z = x^2 + y^2$, so the intersection is the circle at $z = 2$ where $x^2 + y^2 = 2$.

Converting to cylindrical coordinates, this gives us $r^2 = 2$. So we have r ranging from 0 to $\sqrt{2}$, with θ ranging from 0 to 2π, and then z ranges from $x^2 + y^2$ to $\sqrt{6 - x^2 - y^2}$, which is really r^2 to $\sqrt{6 - r^2}$. This gives

$$\begin{aligned}
\iiint_R 1\, dV &= \int_0^{2\pi} \int_0^{\sqrt{2}} \int_{r^2}^{\sqrt{6-r^2}} r \, dz \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^{\sqrt{2}} r\sqrt{6 - r^2} - r^3 \, dr \, d\theta \\
&= \int_0^{2\pi} \frac{-1}{3}(6 - r^2)^{3/2} - \frac{r^4}{4}\Big|_0^{\sqrt{2}} d\theta \\
&= \int_0^{2\pi} \frac{-1}{3} 4^{3/2} - 1 - \frac{-1}{3} 6^{3/2} \, d\theta \\
&= \int_0^{2\pi} -11/3 + \sqrt{24}\, d\theta \\
&= (\sqrt{24} - 11/3) 2\pi.
\end{aligned}$$

Topic 9: Calculus of Curves

(a) Find a parametric equation for a particle moving in a straight line, starting at $(0, 0, 0)$ and moving towards $(3, 2, 1)$.

(b) Suppose another particle follows the path $\vec{r}_2(t) = (t^2, 9-t, t)$. Does this particle's path intersect the path of the particle from part (a)?

(c) Find an equation for the line tangent to the curve $\vec{r}(t) = (3t, \ln(t^2+1), 5t^2+2)$ at the time $t = 3$.

Solution:

(a) There are many correct answers, but one is $\vec{r}_1(t) = (3t, 2t, t)$.

(b) The paths intersect if and only if

$$3t_1 = t_2^2 \qquad 2t_1 = 9 - t_2 \qquad t_1 = t_2$$

The last equation tells us the times must be the same; then the first equation gives us that $t_2 = 3$ and the second equation also gives us that $t_2 = 3$. Thus they cross paths at $t_1 = t_2 = 3$.

(c) We have $\vec{r}'(t) = (3, \frac{2t}{t^2+1}, 10t)$ and thus $\vec{r}'(3) = (3, 6/10, 30)$. So a parametric equation for the line is

$$T(t) = (9 + 3t, \ln(10) + 6t/10, 47 + 30t).$$