Math 2233 Fall 2021
Multivariable Calculus Mastery Quiz 7
Due Thursday, November 4

This week's mastery quiz has topics. **Submit no more than three**. If you already have a 2/2 on a topic, you should not submit it. This week will be the last week Topic 7 is on the quiz.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in at class/recitation on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Topic 7: Multivariable Integrals
- Topic 8: Integrals in Other Coordinates
- Topic 9: Calculus of Curves
- Topic 10: Change of Variables

Name:

Recitation Section:

Topic 7: Multivariable integrals

(a) Find the volume of the solid bounded above by $f(x,y) = e^{-x^2}$, below by the plane $z = 0$, and over the triangle formed by $x = 1, y = 0, y = x$.

Solution:

We have $\int_0^1 \int_0^x e^{-x^2}\,dy\,dx$. Then we compute

$$\begin{aligned}\int_0^1 \int_0^x e^{-x^2}\,dy\,dx &= \int_0^1 ye^{-x^2}\Big|_0^x dx \\ &= \int_0^1 xe^{-x^2}\,dx \\ &= \frac{-1}{2}e^{-x^2}\Big|_0^1 = -\frac{1}{2}(e^{-1}-1) = \frac{1}{2} - \frac{1}{2e}.\end{aligned}$$

We could set it up in the other order: we get $\int_0^1 \int_y^1 e^{-x^2}\,dx\,dy$. But there's no closed-form antiderivative for e^{-x^2}, so we get kinda stuck.

(b) Set up (but do not evaluate) an integral to compute the mass of the region inside the cylinder $y^2 + z^2 = 1$ between the planes $x + y = 3$ and $x - z = 7$, if the density is $\delta(xyz) = e^{-x-yz}$.

Solution:

We can take y from -1 to 1, and then z from $-\sqrt{1-y^2}$ to $\sqrt{1-y^2}$. So we just need to figure out the bounds on x. But we have $x = 7 + z$ on one side and $x = 3 - y$ on the other, so we get the integral

$$\int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{3-y}^{7+z} e^{-x-yz}\,dx\,dz\,dy.$$

Topic 8: Integrals in Other Coordinate Systems

We want to integrate the function $f(x,y,z) = (x^2+y^2+z^2)^{3/2}$, over the region enclosed by the cone $z = \sqrt{3x^2+3y^2}$ and the sphere $x^2+y^2+z^2 = 16$. Set up three different iterated integrals to compute this, in cartesian, cylindrical, and spherical coordinates. Choose one of the integrals you set up and evaluate it.

Solution:

The two surfaces intersect when $x^2+y^2+3x^2+3y^2 = 16$, so $x^2+y^2 = 4$ and we have a circle of radius 2. Thus we get

$$\int_{-2}^2 \int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} \int_{\sqrt{3x^2+3y^2}}^{\sqrt{16-x^2-y^2}} (x^2+y^2+z^2)^{3/2}\,dz\,dy\,dx.$$

I do not want to do this integral.

Cylindrical looks a bit better. We worked out above that the radius of the circle of intersection is 2; and theta goes all the way around, from 0 to 2π. Since $x^2 + y^2 = r^2$ we get a simpler function, and not forgetting the Jacobian we have

$$\int_0^2 \int_0^{2\pi} \int_{\sqrt{3}r}^{\sqrt{16-r^2}} (r^2 + z^2)^{3/2} \cdot r \, dz \, d\theta \, dr.$$

But I still don't want to do that.

In spherical things look pretty okay! We have θ going all the way around from 0 to 2π, and in each direction ρ varies from 0 to 4. We just need to figure out the bounds on ϕ, which is the interior angle of the cone. Since the line has slope $\sqrt{3}$ we can compute this as $\arctan(1/\sqrt{3}) = \pi/6$. Or we could just recognize that this is a 30-60-90 triangle, which gives an interior angle of 30 degrees. Even better, our function is just ρ^3. Thus we have

$$\begin{aligned}
\int_0^4 \int_0^{2\pi} \int_0^{\pi/6} \rho^3 \cdot \rho^2 \sin\phi \, d\phi \, d\theta \, d\rho &= \int_0^4 \int_0^{2\pi} \rho^5 (-\cos\phi) \Big|_0^{\pi/6} \, d\theta \, d\rho \\
&= \int_0^4 \int_0^{2\pi} \rho^5 (1 - \sqrt{3}/2) \, d\theta \, d\rho \\
&= (1 - \sqrt{3}/2) \int_0^4 \rho^5 \theta \Big|_0^{2\pi} \, d\rho \\
&= 2\pi(1 - \sqrt{3}/2) \int_0^4 \rho^5 \, d\rho \\
&= \pi(2 - \sqrt{3}) \frac{\rho^6}{6} \Big|_0^4 \\
&= \pi(2 - \sqrt{3}) 4^6/6 = 2048\pi(2 - \sqrt{3})/3.
\end{aligned}$$

Topic 9: Calculus of Curves

(a) Find a parametric equation for a particle moving in a straight line, starting at $(5, 2, -2)$ and moving towards $(4, -3, 2)$.

(b) Suppose another particle follows the path $\vec{r}_2(t) = (t - 3, 5 - t, t)$. Does this particle's path intersect the path of the particle from part (a)?

(c) Find a (parametric) equation for the line tangent to the curve $\vec{r}(t) = (t^3, \frac{1}{t+1}, t - 2)$ at the time $t = 1$.

Solution:

(a) There are many correct answers, but one is $\vec{r}_1(t) = (5 - t, 2 - 5t, -2 + 4t)$.

(b) The paths intersect if and only if

$$\begin{aligned}
5 - t_1 &= t_2 - 3 \\
2 - 5t_1 &= 5 - t_2 \\
-2 + 4t_1 &= t_2
\end{aligned}$$

The last equation tells us that $t_2 = 4t_1 - 2$; substituting that into the first equation gives $5 - t_1 = 4t_1 - 5$ and thus $t_1 = 2$, which implies $t_2 = 6$. Plugging these numbers into the second equation gives $-8 = -1$, which is false, so the paths do not intersect.

(c) We have $\vec{r}'(t) = (3t^2, \frac{-1}{(t+1)^2}, 1)$ and thus $\vec{r}'(1) = (3, -1/4, 1)$. We compute $\vec{r}(1) = (1, 1/2, -1)$. So a parametric equation for the line is

$$T(t) = (1 + 3t, 1/2 - t/4, -1 + t).$$

Topic 10: Change of Variables

Let R be the parallelogram with vertices $(0,0), (1,2), (3,3), (4,5)$. Find a transformation that translates to the square with vertices $(0,0), (0,1), (1,0), (1,1)$. Use this transformation to compute $\iint_R xy\, dA$.

Solution:

We want $s = 1, t = 0$ to give us the point $(1,2)$, and $s = 0, t = 1$ to give us the point $(3,3)$. So we take $x = s + 3t$ and $y = 2s + 3t$. Then if $s = 1, t = 1$ we get $(4,5)$, and this gives us our mapping of the square to the parallelogram.

We compute the Jacobian

$$\left|\frac{\partial(x,y)}{\partial(s,t)}\right| = \left|\frac{\partial x}{\partial s}\frac{\partial y}{\partial t} - \frac{\partial x}{\partial t}\frac{\partial y}{\partial s}\right| = |1 \cdot 3 - 3 \cdot 2| = 3.$$

Then

$$\begin{aligned}
\iint_R xy\, dA &= \int_0^1 \int_0^1 (s+3t)(2s+3t) \cdot 3\, ds\, dt \\
&= 3 \int_0^1 \int_0^1 2s^2 + 9st + 9t^2\, ds\, dt \\
&= 3 \int_0^1 2s^3/3 + 9s^2t/2 + 9t^2 s\Big|_0^1 dt \\
&= 3 \int_0^1 2/3 + 9t/2 + 9t^2\, dt \\
&= 3\left(2t/3 + 9t^2/4 + 3t^3\right)\Big|_0^1 \\
&= 3(2/3 + 9/4 + 3) = 11 + 27/4 = 17 + 3/4 = 71/4.
\end{aligned}$$