Math 2233 Fall 2021
Multivariable Calculus Mastery Quiz 8
Due Thursday, November 18

This week's mastery quiz has five topics. **Submit no more than four**. If you already have a 2/2 on a topic, you should not submit it. This week will be the last week Topics 8 and 9 are on the quiz.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in at class/recitation on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Topic 8: Integrals in Other Coordinates
- Topic 9: Calculus of Curves
- Topic 10: Change of Variables
- Topic 11: Line Integrals
- Topic 12: Conservative Vector Fields

Name:

Recitation Section:

Topic 8: Integrals in Other Coordinate Systems

We want to find the volume of the region enclosed by the portion of the cylinder $x^2 + y^2 = 9$ with $y \leq 0, z \geq 0$, and the sphere $x^2+y^2+z^2 = 25$. Set up three different iterated integrals to compute this, in cartesian, cylindrical, and spherical coordinates. Choose one of the integrals you set up and evaluate it.

Solution:

The two surfaces intersect when $9 + z^2 = 25$, so $z = \pm 4$ and we have a circle of radius 3. Thus we get

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{0} \int_{0}^{\sqrt{25-x^2-y^2}} 1 \, dz \, dy \, dx.$$

I do not want to do this integral.

Cylindrical looks much better. We worked out above that the radius of the circle of intersection is 3; and theta goes from π to 2π to stay in the half-space where y is negative. Not forgetting the Jacobian we have

$$\int_{0}^{3} \int_{\pi}^{2\pi} \int_{0}^{\sqrt{25-r^2}} 1 \cdot r \, dz \, d\theta \, dr.$$

This one is very doable! We compute

$$\begin{aligned} I &= \int_0^3 \int_\pi^{2\pi} r\sqrt{25 - r^2} \, d\theta \, dr \\ &= \pi \int_0^3 r\sqrt{25-r^2} \, dr \\ &= -\frac{\pi}{3}(25 - r^2)^{3/2} \Big|_0^3 \\ &= \frac{-\pi}{3}\left(16^{3/2} - 25^{3/2}\right) \\ &= \frac{-\pi}{3}(64 - 125) = \frac{61\pi}{3}. \end{aligned}$$

In spherical things don't work out quite so well. The equation of the top is easy, but the side wall of the cylindar is hard to handle.

We again have θ going from π to 2π, and in each direction ρ varies from 0 to 4. But we have to split our other bounds into two parts. When ϕ is between 0 and arctan(3/4), then ρ goes from 0 to 5. But then when ϕ is between arctan(3/4) and $\pi/2$, we need to do some trigonometry: we get $\rho = 3/\sin(\phi)$. Then our integral is

$$\int_{\pi}^{2\pi} \int_{0}^{\arctan(3/4)} \int_{0}^{5} 1 \cdot \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta + \int_{\pi}^{2\pi} \int_{\arctan(3/4)}^{\pi} \int_{0}^{3/\sin(\phi)} \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$

I don't want to do this one either.

Topic 9: Calculus of Curves

(a) Suppose two particles follow the paths $\vec{r}_1(t) = (3t, t^2, 4-t)$ and $\vec{r}_2(t) = (t^2+2, 6-t, 3t-4)$. Do the two particles collide? Do their paths intersect?

(b) Find a (parametric) equation for the line tangent to the curve $\vec{r}(t) = (\sin(t^2), e^t, (t+1)^4 - 2)$ at the time $t = 0$.

Solution:

(a) The particles collide if and only if there's a solution to

$$\begin{aligned} 3t &= t^2 + 2 \\ t^2 &= 6 - t \\ 4 - t &= 3t - 4 \end{aligned}$$

The last equation tells us that $t = 2$. We can check this in the two other equations, and indeed $3 \cdot 2 = 2^2 + 2$ and $2^6 = 6 - 2$. So the particles do collide. (And thus their paths also intersect.)

(b) We have $\vec{r}'(t) = (2t\cos(t^2), e^t, 4(t+1)^3)$ and thus $\vec{r}'(1) = (0, 1, 4)$. We compute $\vec{r}(1) = (0, 1, -1)$. So a parametric equation for the line is

$$T(t) = (0, 1+t, -1+4t).$$

Topic 10: Change of Variables

Use the change of variables $s = y, t = y - x^2$ to evaluate $\iint_R x\,dA$ over the region in the first quadrant bounded by $y = 0, y = 36, y = x^2, y = x^2 - 1$.

Solution:

We have $y = s$ and $x = \sqrt{y-t} = \sqrt{s-t}$, so we compute

$$\left|\frac{\partial(x,y)}{\partial(s,t)}\right| = \begin{vmatrix} \frac{1}{2\sqrt{s-t}} & \frac{-1}{2\sqrt{s-t}} \\ 1 & 0 \end{vmatrix} = \left|\frac{1}{2\sqrt{s-t}}\right| = \frac{1}{2\sqrt{s-t}}.$$

We see that s varies from 0 to 36, and t varies from -1 to 0. The integrand is $x = \sqrt{s-t}$, so we get

$$\begin{aligned} \iint_R x\,dA &= \int_0^{36}\int_{-1}^{0} \sqrt{s-t}\frac{1}{2\sqrt{s-t}}\,dt\,ds \\ &= \int_0^{36}\int_{-1}^{0} \frac{1}{2}\,dt\,ds \\ &= \int_0^{36} \frac{1}{2}\,ds = 18. \end{aligned}$$

Topic 11: Line Integrals

(a) Compute the line integral $\int_C f(x,y)\,ds$ for $f(x,y) = xy$ where C is the first quarter of the unit circle (starting at $(1,0)$ and ending at $(0,1)$).

Solution:

We have $\vec{r}(t) = (\cos(t), \sin(t))$ for $t \in [0, \pi/2]$, so

$$\begin{aligned}
\int_C f\,ds &= \int_0^{\pi/2} f(\vec{r}(t)) \cdot \|\vec{r}'(t)\|\,dt \\
&= \int_0^{\pi/2} \cos(t)\sin(t)\|\langle -\sin(t), \cos(t)\rangle\|\,dt \\
&= \int_0^{\pi/2} \cos(t)\sin(t)\sqrt{\sin^2(t) + \cos^2(t)}\,dt \\
&= \int_0^{\pi/2} \cos(t)\sin(t)\,dt \\
&= \frac{1}{2}\sin^2(t)\Big|_0^{\pi/2} = \frac{1}{2}(\sin^2(\pi/2) - \sin^2(0)) = \frac{1}{2}\cdot 1^2 = 1/2.
\end{aligned}$$

(b) Evaluate the integral $\int_C xy\,dx + yz\,dy + zx\,dz$ around the triangle with vertices $(1,0,0), (0,1,0), (0,0,1)$, oriented in that order.

Solution:

This is a sum of three integrals. We have

$$\begin{aligned}
\vec{r}_1(t) &= (1-t, t, 0) & \vec{r}_1'(t) &= (-1, 1, 0) \\
\vec{r}_2(t) &= (0, 1-t, t) & \vec{r}_2'(t) &= (0, -1, 1) \\
\vec{r}_3(t) &= (t, 0, 1-t) & \vec{r}_3'(t) &= (1, 0, -1)
\end{aligned}$$

Then we compute

$$\begin{aligned}
\int_{\vec{r}_1} xy\,dx + yz\,dy + zx\,dz &= \int_0^1 (1-t)(t)(-1) + t\cdot 0(1) + 0(1-t)0\,dt \\
&= \int_0^1 t^2 - t\,dt = \frac{t^3}{3} - \frac{t^2}{2}\Big|_0^1 = 1/3 - 1/2 = -1/6 \\
\int_{\vec{r}_2} xy\,dx + yz\,dy + zx\,dz &= \int_0^1 0(1-t)0 + (1-t)(t)(-1) + t(0)(1)\,dt \\
&= \int_0^1 t^2 - t\,dt = -1/6 \\
\int_{\vec{r}_3} xy\,dx + yz\,dy + zx\,dz &= \int_0^1 t(0)(1) + 0(1-t)(0) + (1-t)(t)(-1)\,dt \\
&= \int_0^1 t^2 - t\,dt = -1/6 \\
\int_C xy\,dx + yz\,dy + zx\,dz &= -1/6 - 1/6 - 1/6 = -1/2.
\end{aligned}$$

Topic 12: Conservative Vector Fields

(a) Find a potential field for $\vec{F}(x,y) = (x+y)\vec{i} + (x-y)\vec{j}$ or prove none exists.

Solution:

If $\vec{F} = \nabla f$, we must have

$$f(x,y) = \frac{1}{2}x^2 + xy + g(y)$$
$$f(x,y) = xy - \frac{1}{2}y^2 + h(x)$$

and we can satisfy this with

$$f(x,y) = x^2/2 + xy - y^2/2.$$

(b) Let $f(x,y,z) = xy + zx^2$. Compute $\int_C \nabla f \, ds$ where C is parametrized by the curve $\vec{r}(t) = (2t-1, t^4+t, \sin(\pi t))$ for $t \in [0,2]$.

Solution:

We could compute the whole line integral out, but that would be awful. Instead we use the fundamental theorem of line integrals, and we have

$$\begin{aligned}\int_C \nabla f \, ds &= f(r(2)) - f(r(0)) \\ &= f(3,18,0) - f(-1,0,0) \\ &= 54 + 0 - (0+0) = 54.\end{aligned}$$