Math 2233 Fall 2021
Multivariable Calculus Mastery Quiz 9
Due Thursday, December 2

This week's mastery quiz has five topics. **Submit no more than four**. If you already have a 2/2 on a topic, you should not submit it. This week will be the last week Topic 10 is on the quiz.

Feel free to consult your notes or speak to me privately, but please don't talk about the actual quiz questions with other students in the course or post about it publicly.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please turn this quiz in at class/recitation on Wednesday. You may print this document out and write on it, or you may submit your work on separate paper; in either case make sure your name and recitation section are clearly on it. If you absolutely cannot turn it in in person, you can submit it electronically but this should be a last resort.

Topics on This Quiz

- Topic 10: Change of Variables
- Topic 11: Line Integrals
- Topic 12: Conservative Vector Fields
- Topic 13: Surface Integrals
- Topic 14: Green's and Stokes's

Name:

Recitation Section:

Topic 10: Change of Variables

Use a change of variables to integrate the function $(x+y)(x-y)$ over the diamond bounded by $x+y=2, x+y=0, x-y=0, x-y=2$.

Solution: We want to reparametrize this with $s=x+y, t=x-y$. Our bounds are $t=2, s=0, t=2, t=0$, and our function is st.

To compute the Jacobian we need to find x and y as a function of s and t. We see that $s+t=2x$ so $x=\frac{s+t}{2}$; similarly, $s-t=2y$ so $y=\frac{s-t}{2}$. Then the Jacobian is

$$\frac{\partial(x,y)}{\partial(s,t)} = \begin{vmatrix} \frac{\partial x}{\partial s} & \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial x} & \frac{\partial y}{\partial t} \end{vmatrix} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -1/4 - 1/4 = -1/2$$

so $\left|\frac{\partial(x,y)}{\partial(s,t)}\right| = 1/2$. Then the integral is

$$\begin{aligned}\int_R (x+y)(x-y)\,dA &= \int_0^2 \int_0^2 st/2\,ds\,dt \\ &= \int_0^2 s^2t/4|_0^2\,dt = \int_0^2 t\,dt \\ &= t^2/2|_0^2 = 2.\end{aligned}$$

Topic 11: Line Integrals

(a) Compute the arc length of the curve $\vec{r}(t) = (t^2, 3, \frac{1}{3}t^3)$ between the points $(0,3,0)$ and $(1,3,1/3)$.

Solution:

We know the arc length of a curve is $\int_C 1\,ds = \int_a^b \|\vec{r}'(t)\|\,dt$. We see that our two points are $\vec{r}(0)$ and $\vec{r}(1)$, so we have

$$\begin{aligned}L &= \int_0^1 \|(2t,0,t^2)\|\,dt \\ &= \int_0^1 \sqrt{4t^2+t^4}\,dt = \int_0^1 t\sqrt{4+t^2}\,dt \\ &= \frac{1}{3}(4+t^2)^{3/2}\Big|_0^1 = \frac{1}{3}(5^{3/2}-4^{3/2}) = \frac{1}{3}(\sqrt{125}-8).\end{aligned}$$

(b) Let C be the helix that winds around the cylinder $x^2+y^2=1$ (counterclockwise viewed from the positive z-axis looking down on the xy-plane), starting at $(1,0,0)$, winding around the cylinder once, and ending at the point $(1,0,1)$.

Compute the line integral of the vector field $\vec{F}(x,y,z) = (-y, x, z^2)$. (Hint: this vector field is **not** conservative.)

Solution:

We parametrize the curve with $\vec{r}(t) = (\cos(t), \sin(t), \frac{t}{2\pi})$. Then we have

$$\begin{aligned}\int \vec{F} \cdot d\vec{r} &= \int_0^{2\pi} \left(-\sin(t), \cos(t), \frac{t^2}{4\pi^2}\right) \cdot \left(-\sin(t), \cos(t), \frac{1}{2\pi}\right) dt \\ &= \int_0^{2\pi} \sin^2(t) + \cos^2(t) + \frac{t^2}{8\pi^3}\, dt \\ &= \int_0^{2\pi} 1 + \frac{t^2}{8\pi^3}\, dt \\ &= t + \frac{t^3}{24\pi^3}\Big|_0^{2\pi} = 2\pi + \frac{8\pi^3}{24\pi^3} = 2\pi + \frac{1}{3}.\end{aligned}$$

Topic 12: Conservative Vector Fields

(a) Find a potential field for $\vec{F}(x, y, z) = y\vec{i} + x\vec{j} + xz\vec{k}$ or prove none exists.

Solution: There is no potential field, and we can prove this in two different ways.

First, if $\vec{F} = \nabla f$, we must have

$$\begin{aligned} f(x, y, z) &= xy + g_1(y, z) \\ f(x, y, z) &= xy + g_2(x, z) \\ f(x, y, z) &= xz^2/2 + g_3(x, y) \end{aligned}$$

So we need a function that is equal to xy plus a function purely of y and z, but also equal to $xz^2/2$ plus a function purely of x and y, and you can't have both of those things at once.

Alternatively, we can just compute the curl of $\vec{F}$:

$$\begin{aligned}\nabla \times \vec{F}(x, y, z) &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y & x & xz \end{vmatrix} \\ &= 0\vec{i} - z\vec{j} + (1 - 1)\vec{k} = -z\vec{j} \neq \vec{0}.\end{aligned}$$

Since the curl is not $\vec{0}$ we know this must not be a conservative field.

(b) Let $f(x, y, z) = 3x^2 y + \frac{z}{x}$. Compute $\int_C \nabla f\, ds$ where C is parametrized by the curve $\vec{r}(t) = (t, \frac{t^2}{t+4}, t)$ for $t \in [1, 4]$.

Solution:

We could compute the whole line integral out, but that would be awful. Instead we use the fundamental theorem of line integrals, and we have

$$\begin{aligned}\int_C \nabla f\, ds &= f(r(4)) - f(r(1)) \\ &= f(4, 2, 4) - f(1, 1/5, 1) \\ &= 96 + 1 - (3/5 + 1) = 96 - \frac{3}{5} = 95 + \frac{2}{5} = \frac{477}{5} = 95.4\end{aligned}$$

Topic 13: Surface Integrals

(a) Find the surface area of the part of the paraboloid $z = 16 - x^2 - y^2$ which lies above the xy-plane.

Solution:

We want to compute $\int_S 1\, dS = \iint_R \|\vec{r}_s \times \vec{r}_t\|\, ds\, dt$. We know from class that if our surface is the graph of a function, then

$$\|\vec{r}_s \times \vec{r}_t\| = \sqrt{1 + (f_x)^2 + (f_y)^2} = \sqrt{1 + 4x^2 + 4y^2}.$$

We're integrating over the circle $x^2 + y^2 = 16$, so we should use polar coordinates. So we get the integral

$$\begin{aligned}\int_0^4 \int_0^{2\pi} \sqrt{1+4r^2}r\, d\theta\, dr &= \int_0^4 2\pi r\sqrt{1+4r^2}\, dr \\ &= \frac{\pi}{6}(1+4r^2)^{3/2}\Big|_0^4 = \frac{\pi}{6}(65^{3/2} - 1).\end{aligned}$$

(b) Compute the flux of the vector field $\vec{F}(x, y, z) = y\vec{i} + z\vec{k}$ upwards through portion of the cone $z = 1 - \sqrt{x^2 + y^2}$ above the plane $z = 0$.

Solution:

We can parametrize the cone with $(r\cos(t), r\sin(t), 1 - r)$, where $0 \le r \le 1$ and $0 \le t \le 2\pi$, and we get Jacobian term

$$\begin{aligned}\vec{r}_r \times \vec{r}_t &= (\cos(t), \sin(t), -1) \times (-r\sin(t), r\cos(t), 0) \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \cos(t) & \sin(t) & -1 \\ -r\sin(t) & r\cos(t) & 0 \end{vmatrix} = r\cos(t)\vec{i} + r\sin(t)\vec{j} + (r\cos^2 t + r\sin^2 t)\vec{k} \\ &= r\cos(t)\vec{i} + r\sin(t)\vec{j} + r\vec{k}.\end{aligned}$$

This is the right orientation since it is in fact pointed upwards for positive r.

Then we compute flux as

$$\begin{aligned}\int_S \vec{F} \cdot d\vec{S} &= \int_0^1 \int_0^{2\pi} (r\sin(t), 0, 1-r) \cdot (r\cos(t), r\sin(t), r)\, dt\, dr \\ &= \int_0^1 \int_0^{2\pi} r^2 \sin(t)\cos(t) + r - r^2\, dt\, dr \\ &\int_0^1 \frac{1}{2}r^2\sin^2(t) + rt - r^2 t\Big|_0^{2\pi} dr \\ &\int_0^1 2\pi r - 2\pi r^2\, dr = \pi r^2 - \frac{2\pi}{3}r^3\Big|_0^1 = \frac{\pi}{3}.\end{aligned}$$

Topic 14: Green's Theorem and Stokes's Theorem

(a) **Use Green's Theorem** to evalute $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y) = x^2y^2\vec{i} + 2xy\vec{j}$ and C is the boundary of the square (clockwise) through points $(0,0), (0,3), (3,3), (3,0)$.

Solution:

We could parametrize the outside of the square, but that would actually suck. Instead we integrate over the square $[0,3]\times[0,3]$, which works by Green's Theorem. We compute the curl

$$\nabla\times\vec{F}(x,y) = \frac{\partial}{\partial x}(2xy) - \frac{\partial}{\partial y}(x^2y^2) = 2y - 2x^2y.$$

So the integral *counterclockwise* around the square is givin by Green's Theorem:

$$\begin{aligned}\int_{-C}\vec{F}\cdot d\vec{r} &= \int_0^3\int_0^3 2y - 2x^2y\,dy\,dx \\ &= \int_0^3 y^2 - x^2y^2\Big|_0^3\,dx \\ &= \int_0^3 9 - 9x^2\,dx = 9x - 3x^3\Big|_0^3 = 27 - 81 = -54.\end{aligned}$$

Thus the integral *clockwise* around the square is 54.

(b) **Use Stokes's Theorem** to compute $\int_C \vec{F}\cdot d\vec{r}$ where $\vec{F}(x,y,z) = (x^2 - y, y^2 + x, 1)$ and C is the intersection of $x^2 + y^2 = 1$ and $z = y^2$ viewed from the point $(0,0,100)$.

Solution:

Instead of parametrizing the curve, we can just find the surface whose boundary is that curve. We can take $\vec{r}(s,t) = (s,t,t^2)$ for $s^2 + t^2 \le 1$, and then we get normal vector

$$\begin{aligned}\vec{r}_s\times\vec{r}_t = (1,0,0)\times(0,1,2t) &= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & 0 & 0\\ 0 & 1 & 2t\end{vmatrix} \\ &= -2t\vec{j} + \vec{k}.\end{aligned}$$

Since we want the curve to be oriented counterclockwise when viewed from above, that means we want the normal vector pointing upwards, which this one does, so this is the correct orientation.

We compute

$$\begin{aligned}\nabla\times\vec{F} &= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ x^2 - y & y^2 + x & 1\end{vmatrix} \\ &= 0\vec{i} + 0\vec{j} + (1+1)\vec{k} = 2\vec{k}.\end{aligned}$$

Thus we have

$$
\begin{aligned}
\int_C \vec{F} \cdot d\vec{S} &= \int_S \nabla \times \vec{F} \cdot \vec{S} \\
&= \int_{-1}^{1} \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} (2\vec{k}) \cdot (-2t\vec{k} + \vec{k}) \, dt \, ds \\
&= \int_{-1}^{1} \int_{-\sqrt{1-s^2}}^{\sqrt{1-s^2}} 2 \, dt \, ds.
\end{aligned}
$$

At this point we could convert this integral to polar coordinates, but we can also observe that we're just integrating the number 2 over the unit circle, which has area π. So the total integral here is 2π.