Taylor Series

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

$\frac{1}{1-x}$ very limited

$$f(x) = x + x^2$$
$$= 0 + x + x^2 + 0x^3 + \cdots$$

want a PS for e^x

Suppose $e^x = \sum_{n=0}^{\infty} c_n x^n$

$$= \overset{1}{c_0} + \overset{1}{c_1} x + \overset{1/2}{c_2} x^2 + \overset{1/6}{c_3} x^3 + \cdots$$

$$1 = e^0 = c_0$$

$$e^x = \frac{d}{dx} \sum_{n=0}^{\infty} c_n x^n$$

$$= 0 + \overset{1}{c_1} + 2\overset{1/2}{c_2} x + 3\overset{1/6}{c_3} x^2 + 4c_4 x^3 + \cdots$$

$$1 = e^0 = c_1$$

$$e^x = \frac{d^2}{dx^2} \sum c_n x^n = 0 + 0 + 2\overset{1/2}{c_2} + 6\overset{1/6}{c_3} x + 12 c_4 x^2$$

$$1 = e^0 = 2c_2$$

$$e^x = \frac{d^3}{dx^3} \sum c_n x^n = 0 + 0 + 0 + 6\overset{1/6}{c_3} + 24 c_4 x + \cdots$$

$$1 = e^0 = 6c_3$$

$c_0 = 1$
$c_1 = 1$
$c_2 = 1/2$
$c_3 = 1/6$
$c_4 = 1/24$
$c_5 = 1/120$
$c_n = n!$

$y' = y$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$$

$$e^x \approx \sum_{n=0}^{k} \frac{x^n}{n!}$$

Thm: If f has PS rep at a,

i.e. if $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$

Then $c_n = \dfrac{f^{(n)}(a)}{n!}$.

$c_0 + c_1 (x-a) + c_2 (x-a)^2 + \cdots$

$f^{(0)}(a) = f(a)$

$f^{(1)}(a) = f'(a)$

Dfn: The <u>Taylor Series</u> of f centered at a is

$$T_f(x,a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{6}(x-a)^3 + \cdots$$

The Maclaurin series of f is $T_f(x,0)$.

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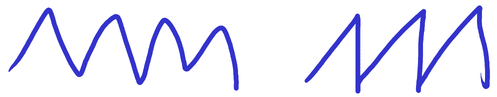
1) If fn doesn't have ∞ derivatives, no Taylor series.



 , $\sqrt[3]{x}$

2) Some functions <u>have</u> TS, but aren't = to them.

The nth Taylor Polynomial of f at a is

$$T_{f,n}(x,a) = T_n(x,a) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k.$$

Dfn: define $R_n(x,a) = |f(x) - T_n(x,a)|$

Want

$$f(x) = \lim_{n\to\infty} T_n(x,a)$$

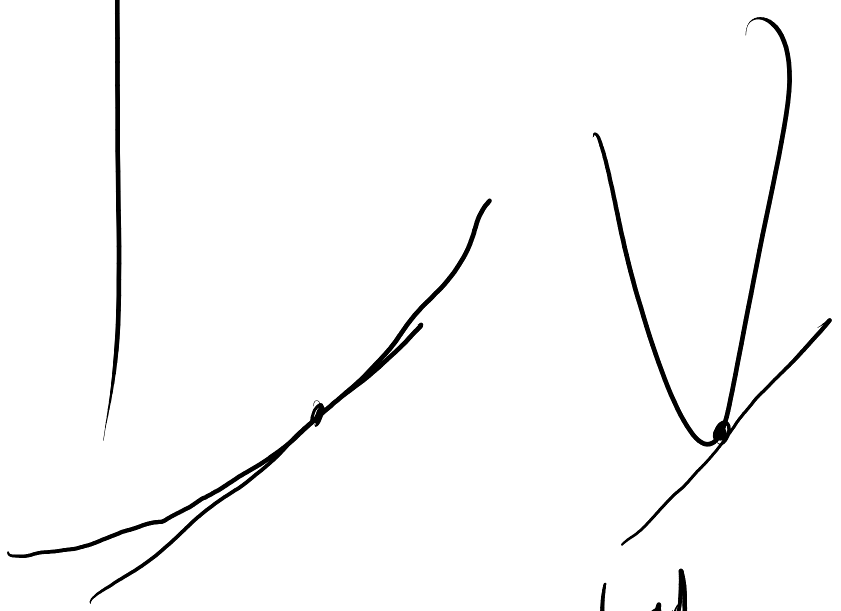
$$\lim_{n\to\infty} f(x) - T_n(x,a) = 0$$

$$\lim_{n\to\infty} R_n(x,a) = 0.$$

Prop: If f has enough derivatives on an interval containing a, then there is a number z btwn x and a s.t.

$$R_n(x,a) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}.$$

$$f(x) - f(a) = \frac{f'(z)}{1}(x-a)$$



good

bad

Ex: show $e^x = T(x,0)$

$$\frac{d^{n+1}}{dx^{n+1}} e^x = e^x$$

$$R_n(x,0) = \frac{e^z}{(n+1)!} x^{n+1}$$

$$= \frac{x^{n+1}}{(n+1)!} e^z$$

if $x>0$

$$\le \frac{x^{n+1}}{(n+1)!} e^x$$

$$0 \le R_n(x,0) \le \frac{x^{n+1}}{(n+1)!} e^x$$

$$\lim_{n\to\infty} \frac{x^{n+1}}{(n+1)!} e^x = 0$$

So by squeeze thm,

$$\lim_{n\to\infty} R_n(x,0) = 0.$$

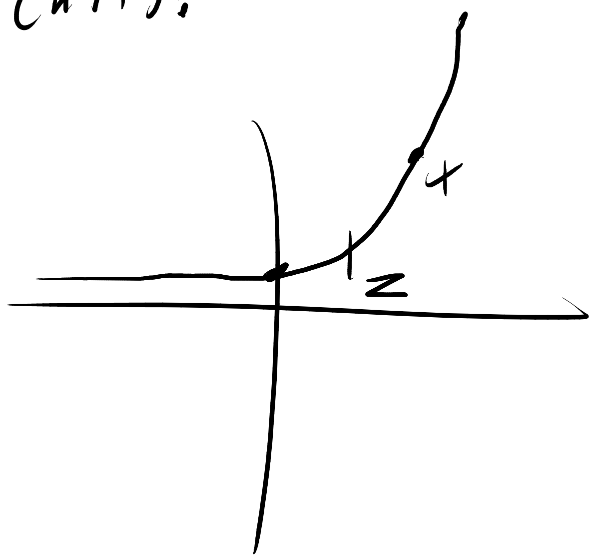
approximate e?

$$e^1 = \sum_{n=0}^{\infty} \frac{1^n}{n!}$$

$$= 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \cdots$$

there is a number z btwn x and a

$$R_n(x,a) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}.$$



find $T_{e^x}(x, 1)$

$f^{(n)}(1) = e^1 = e$

$$T(x,1) = \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n$$

$$= \sum_{n=0}^{\infty} \frac{e^1}{n!}(x-1)^n$$

$$= e + e(x-1) + \frac{e}{2}(x-1)^2$$

if $x > 1$

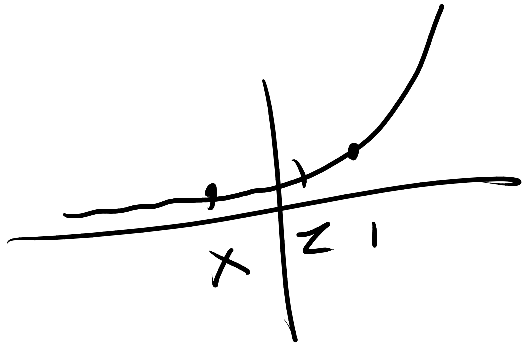
$$R_n(x,1) = \frac{e^z}{(n+1)!}(x-1)^{n+1}$$

$$\leq \frac{(x-1)^{n+1}}{(n+1)!} e^x \to 0$$

If $x < 1$

$$R_n(x,1) = \frac{e^z}{(n+1)!}(x-1)^{n+1}$$

$$\leq \frac{(x-1)^{n+1}}{(n+1)} e^1 \to 0$$



$e^x =$

$$T(x,1) = \sum_{n=0}^{\infty} \frac{e}{n!}(x-1)^n$$

$$= e\sum_{n=0}^{\infty} \frac{1}{n!}(x-1)^n$$

$$= e \cdot e^{(x-1)}$$

$$= e^{1+(x-1)} = e^x.$$

$$f(x) = \ln(1+x)$$

$$f' = \frac{1}{1+x}$$

$$f'' = \frac{-1}{(1+x)^2}$$

$$f''' = \frac{2}{(1+x)^3}$$

$$f'''' = \frac{-6}{(1+x)^4}$$

$$f^{(n)}(x) = \frac{(n-1)!\,(-1)^{n-1}}{(1+x)^n}$$

$$f^{(n)}(0) = \frac{(n-1)!\,(-1)^{n-1}}{\cancel{(1+0)^n}}$$

$$T_f(x,0) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}(x-0)^n$$

$$= \sum_{n=0}^{\infty} \frac{(n-1)!\,(-1)^{n-1}}{n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{n} x^n.$$

This is exactly what we got by integrating a geo series.

$$\frac{(n-1)!}{n!}$$

$$R_n = \frac{f^{(n+1)}(z)}{(n+1)!}(x)^n$$

$$= \frac{n!\,.\,(-1)^n}{(n+1)!\,(1+z)^n}\,x^n$$

$$= \frac{(-1)^n}{n+1}\,\frac{x^n}{(1+z)^n}$$