Math 1232 Final Solutions

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1. This test is due at midnight on Tuesday, May 4. Logistically, this will work just like the mastery quizzes: download it, write up your answers, and upload them to Blackboard for us to grade.
2. You will have three hours for this test. Please write down your start and end times on the test and include that in your upload. You may not spend more than three hours on the test unless you have a specific accommodation.
3. You are not allowed to consult books or notes during the test, but you may use a two-page cheat sheet you have made for yourself ahead of time. Please upload your sheet along with your test.
4. If you have questions, I will be online and responsive during the scheduled exam time. If you want to take the test at a time you know I'll be able to answer any questions quickly, I encourage you to use that time slot.
5. You may use a calculator, but don't use a graphing calculator or anything else that can do symbolic computations. Using a calculator for basic arithmetic is fine.

Name:

Time Started:

Time Completed:

Problem 1.

(a) $\int x\cos(4x)\,dx =$

Solution:

We use integration by parts, taking $u = x, du = dx, dv = \cos(4x)\,dx, v = \sin(4x)/4$. Then

$$\begin{aligned}\int x\cos(4x)\,dx &= \int u\,dv = uv - \int v\,du \\ &\frac{x\sin(4x)}{4} - \int \frac{\sin(4x)}{4}\,dx \\ &= \frac{x\sin(4x)}{4} + \frac{\cos(4x)}{16} + C.\end{aligned}$$

(b) $\int_{-\pi/4}^{\pi/4} \sec^4(x)\tan^2(x)\,dx =$

Solution: We have an even number of secants, so we can reduce to two:

$$\begin{aligned}\int_{-\pi/4}^{\pi/4} \sec^4(x)\tan^2(x)\,dx &= \int_{-\pi/4}^{\pi/4} \sec^2(x)\tan^2(x)(\tan^2(x)+1)\,dx \\ &= \int_{-\pi/4}^{\pi/4} \sec^2(x)\tan^4(x) + \sec^2(x)\tan^2(x)\,dx \\ &\quad u = \tan(x), du = \sec^2(x)\,dx \\ &= \int_{-1}^{1} u^4 + u^2\,du = \frac{u^5}{5} + \frac{u^3}{3}\Big|_{-1}^{1} = \frac{1}{5} + \frac{1}{3} - \left(\frac{-1}{5} + \frac{-1}{3}\right) = \frac{2}{5} + \frac{2}{3} = \frac{16}{15}.\end{aligned}$$

(c) $\int_{-2}^{2} \frac{1}{(x^2+4)^{3/2}}\,dx =$

Solution: We use trigonometric substitution. Let $x = 2\tan\theta$ so we have $dx = 2\sec^2\theta\,d\theta$. When $x = 2$ then $2\tan\theta = 1$ so $\tan\theta = 1$ and $\theta = \pi/4$, and similarly when $x = -2$ then $\theta = -\pi/4$. Then

$$\begin{aligned}\int_{-2}^{2} \frac{1}{(x^2+4)^{3/2}}\,dx &= \int_{-\pi/4}^{\pi/4} \frac{2\sec^2\theta\,d\theta}{(4\tan^2\theta+4)^{3/2}} \\ &= \int_{-\pi/4}^{\pi/4} \frac{2\sec^2\theta\,d\theta}{(4\sec^2\theta)^{3/2}} \\ &= \frac{2}{4^{3/2}}\int_{-\pi/4}^{\pi/4} \frac{\sec^2\theta\,d\theta}{\sec^3\theta} \\ &= \frac{1}{4}\int_{-\pi/4}^{\pi/4} \cos\theta\,d\theta = \frac{1}{4}\sin\theta\Big|_{-\pi/4}^{\pi/4} = \frac{1}{4}\left(\sin(\pi/4) - \sin(-\pi/4)\right) = \frac{\sqrt{2}}{4}\end{aligned}$$

(d) $\int_{0}^{2} \frac{1}{\sqrt{2-x}}\,dx =$

Solution: The function is discontinuous and has an asymptote at 2, so this is an improper integral:

$$\begin{aligned}\int_{0}^{2} \frac{1}{\sqrt{2-x}}\,dx &= \lim_{t\to 2^-} \int_{0}^{t} \frac{dx}{\sqrt{2-x}} \\ &= \lim_{t\to 2^-} -2\sqrt{2-x}\Big|_{0}^{t} \\ &= \lim_{t\to 2^-} -2\sqrt{2-t} + 2\sqrt{2-0} = 2\sqrt{2}.\end{aligned}$$

Problem 2.

(a) Analyze the convergence of $\displaystyle\sum_{n=1}^{\infty} \frac{(-10)^n}{n!}$

Solution: Since we have a factorial we use the ratio test. We have

$$\lim_{n\to\infty} \left| \frac{(-10)^{n+1}/(n+1)!}{(-10)^n/n!} \right| = \left| \frac{-10}{n+1} \right| = 0$$

So by the ratio test this converges absolutely.

(b) Analyze the convergence of $\displaystyle\sum_{n=1}^{\infty} (-1)^n \frac{n^3 - n + 1}{n^4 - 3n^2 + 4}$.

Solution: First we examine absolute convergence. We use the limit comparison test: we have

$$\lim_{n\to\infty} \frac{(n^3 - n + 1)/(n^4 - 3n^2 + 4)}{1/n} = \lim_{n\to\infty} \frac{n^4 - n^2 + n}{n^4 - 3n^2 + 4} = 1$$

and thus since $\sum \frac{1}{n}$ diverges, so does $\sum \frac{n^3-n+1}{n^4-3n^2+4}$. So the sum does not converge absolutely.

However, this is an alternating series, and the terms are decreasing and tend towards zero. Thus by the alternating series test the series converges conditionally.

(c) Analyze the convergence of $\displaystyle\sum_{n=1}^{\infty} (-1)^n 2ne^{-n^2}$

Solution: This is an alternating series and the terms go to zero, so it must converge by the alternating series test.

To test absolute convergence, we consider $\sum_{n=1}^{\infty} 2ne^{-n^2}$. The terms are positive, continuous, and decreasing, so we can use the integral test.

$$\int_1^{+\infty} 2ne^{-n^2} = \lim_{t\to+\infty} \int_1^t 2ne^{-n^2} = \lim_{t\to+\infty} -e^{-n^2}|_1^t = \lim_{t\to+\infty} -e^{-t^2} + e^{-1^2} = e^{-1}.$$

Since this improper integral converges, the series $\sum_{n=1}^{\infty} 2ne^{-n^2}$ also converges.

Problem 3.

(a) Find the radius and interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{n^3}{3^n} x^n$

Solution: Using the ratio test, we have

$$\lim_{n\to\infty} \left| \frac{(n+1)^3 x^{n+1}/3^{n+1}}{n^3 x^n / 3^n} \right| = \lim_{n\to\infty} \frac{|x|}{3} \frac{(n+1)^3}{n^3} = \frac{|x|}{3}.$$

Thus the series converges for $|x| < 3$ and diverges for $|x| > 3$. We have to check endpoints. When $x = 3$ then the series is $\sum n^3$ which diverges by the divergence test, and when $x = -3$ the series is $\sum (-1)^n n^3$ which also diverges by the divergence test. So the interval of convergence is $(-3, 3)$.

(b) Write a power series for the function $x \ln(1 + x^3)$.

Solution: We know that

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

So we must have

$$x \ln(1 + x^3) = x \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{3n}}{n} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^{3n+1}}{n}.$$

(c) Compute the second-order Taylor polynomial for $f(x) = \sin(x)\cos(x)$ centered at $\pi/4$.

Solution:

$$
\begin{aligned}
f(x) &= \sin(x)\cos(x) & f(\pi/4) &= \frac{1}{2} \\
f'(x) &= \cos^2(x) - \sin^2(x) & f'(\pi/4) &= \frac{1}{2} - \frac{1}{2} = 0 \\
f''(x) &= -2\cos(x)\sin(x) - 2\sin(x)\cos(x) = -4\sin(x)\cos(x) & f''(\pi/4) &= -2
\end{aligned}
$$

and thus

$$
\begin{aligned}
T_2(x, \pi/4) &= f(\pi/4) + \frac{f'(\pi/4)}{1}(x - \pi/4) + \frac{f''(\pi/4)}{2!}(x - \pi/4)^2 \\
&= \frac{1}{2} - (x - \pi/4)^2.
\end{aligned}
$$

Problem 4.

(a) Find an equation for the tangent line to the curve defined by $x = t^2/3, y = \frac{t}{t^2+1}$ at time $t = 3$.

Solution:

We have

$$
\begin{aligned}
x(3) &= 3 \\
y(3) &= \frac{3}{10} \\
\frac{dx}{dt} &= \frac{2t}{3} \\
\frac{dy}{dt} &= \frac{t^2 + 1 - 2t^2}{(t^2+1)^2} = \frac{1 - t^2}{(1 + t^2)^2} \\
\frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{(-8)/100}{2} = \frac{-1}{25} \\
y - \frac{3}{10} &= \frac{-1}{25}(x - 3) \\
\text{(or)} \qquad y &= \frac{-x}{25} + \frac{21}{50}.
\end{aligned}
$$

(b) Find the solution to $y' = x/y^2$ when $y(0) = 3$.

Solution:

$$
\begin{aligned}
\frac{dy}{dx} &= x/y^2 \\
\int y^2\,dy &= \int x\,dx \\
\frac{y^3}{3} &= \frac{x^2}{2} + C \\
y &= \sqrt[3]{3x^2/2 + 3C}.
\end{aligned}
$$

To find the specific solution, we compute

$$
\begin{aligned}
3 &= \sqrt[3]{0 + 3C} \\
27 &= 3C \\
C &= 9.
\end{aligned}
$$

Thus the specific solution to our differential equation is

$$y = \sqrt[3]{3x^2/2 + 27}$$

or, alternatively,

$$\frac{y^3}{3} = \frac{x^2}{2} + 9.$$

(c) Find the arc length of the curve given by $y = \frac{2}{3}x^{3/2} - \frac{\sqrt{x}}{2}$ between $x = 1$ and $x = 4$.

Solution: We have $y' = \sqrt{x} - \frac{1}{4\sqrt{x}}$ and so

$$\begin{aligned} L &= \int_1^4 \sqrt{1 + \left(\sqrt{x} - \frac{1}{4\sqrt{x}}\right)^2}\, dx \\ &\int_1^4 \sqrt{1 + x + \frac{1}{16x} - \frac{1}{2}}\, dx \\ &= \int_1^4 \frac{1}{\sqrt{16x}} \sqrt{16x^2 + 8x + 1}\, dx \\ &= \int_1^4 \frac{4x+1}{\sqrt{16x}}\, dx = \int_1^4 \sqrt{x} + \frac{1}{4\sqrt{x}} \\ &= \frac{2}{3}x^{3/2} + \frac{\sqrt{x}}{2}\Big|_1^4 = \frac{2}{3}\cdot 8 + \frac{2}{2} - \frac{2}{3} - \frac{1}{2} = \frac{14}{3} + \frac{1}{2} = \frac{31}{6}. \end{aligned}$$

Problem 5.

(a) Let $g(x) = \sqrt[4]{3x^5 + 4x^3 + 2x + 7}$ for $x \geq 0$. Find $(g^{-1})'(2)$.

Solution: We see that $g(1) = 2$, so $g^{-1}(2) = 1$. Then by the Inverse Function Theorem we have

$$\begin{aligned} (g^{-1})'(2) &= \frac{1}{g'(g^{-1}(2))} = \frac{1}{g'(1)} \\ g'(x) &= \frac{1}{4}(3x^5 + 4x^3 + 2x + 7)^{-3/4}(15x^4 + 12x^2 + 2) \\ g'(1) &= \frac{1}{4}\cdot\frac{1}{8}(29) = \frac{29}{32} \\ (g^{-1})'(1) &= \frac{32}{29}. \end{aligned}$$

(b) Approximate $\int_0^3 \sin(\pi x/3)\, dx$ using three intervals and the Trapezoidal rule.

Solution:

$$\begin{aligned} \int_0^3 \sin(\pi x/3)\, dx &\approx \frac{\sin(0) + \sin(\pi/3)}{2} + \frac{\sin(\pi/3) + \sin(2\pi/3)}{2} + \frac{\sin(2\pi/3) + \sin(\pi)}{2} \\ &= \frac{1}{2}\left(\frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{2} + \frac{\sqrt{3}}{4}\right) = \sqrt{3}. \end{aligned}$$

(c) Use a fifth-degree Taylor polynomial centered at 0 to apprximate sin(1).

Solution: We know that

$$\begin{aligned} \sin(x) &\approx x - \frac{x^3}{3!} + \frac{x^5}{5!} \\ \sin(1) &\approx 1 - \frac{1}{6} + \frac{1}{120} = \frac{101}{120} \\ &\quad (\approx .841667). \end{aligned}$$