Math 1232 Midterm Solutions

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1. This test is due Tuesday, March 2 at 7 PM. Logistically, this will work just like the mastery quizzes: download it, write up your answers, and upload them to Blackboard for us to grade.
2. You will have two hours for this test. Please write down your start and end times on the test and include that in your upload. You may not spend more than two hours on the test unless you have a specific accommodation.
3. You are not allowed to consult books or notes during the test, but you may use a one-page, two-sided cheat sheet you have made for yourself ahead of time. Please upload your sheet along with your test.
4. If you have questions, I will be online and responsive during the usual class times. If you want to take the test at a time you know I'll be able to answer any questions quickly, I encourage you to use one of those time slots.
5. You may use a calculator, but don't use a graphing calculator or anything else that can do symbolic computations. Using a calculator for basic arithmetic is fine.

Name:

Time Started:

Time Completed:

Problem 1. (a) Let $f(x) = \sqrt{x^5 + 3x^3 + 5x}$. Find $(f^{-1})'(3)$.

Solution:

Plugging in numbers, we see that $f(1) = \sqrt{1+3+5} = \sqrt{9} = 3$, so $f^{-1}(3) = 1$. Then by the Inverse Function Theorem we have $(f^{-1})'(3) = \frac{1}{f'(1)}$. But

$$f'(x) = \frac{1}{2}(x^5 + 3x^3 + 5x)^{-1/2}(5x^4 + 9x^2 + 5)$$
$$f'(1) = \frac{1}{2}(9)^{-1/2}(5 + 9 + 5) = \frac{19}{6}.$$

Thus by the inverse function theorem we have

$$(f^{-1})'(3) = \frac{6}{19}.$$

(b) Find $\lim_{x\to 0} \frac{\sin(x)}{e^{3x} - 1}$.

Solution:

The top and bottom both approach zero, so we can use L'Hospital's Rule.

$$\lim_{x\to 0} \frac{\sin(x)}{e^{3x} - 1} = \lim_{x\to 0} \frac{\cos(x)}{3e^{3x}} = \frac{1}{3}.$$

(c) Compute the following. In all cases your answers should be exact, with no decimals, and no logs or exponentials or trig functions.

$$\log_2(8) + \log_2(6) - \log_2(3)$$

Solution: $3 + \log_2(2) = 4$

$$\arctan(\sqrt{3}) =$$

Solution: $\pi/3$

$$\sin(\arctan(4/3)) =$$

Solution: $4/5$

Problem 2. (a) Compute $f'(x)$ where $f(x) = \arcsin \log_3(x^2)$.

Solution:

$$f'(x) = \frac{1}{\sqrt{1 - \log_3(x^2)^2}} \frac{1}{x^2 \ln(3)} 2x$$

(b) Find the tangent line to $h(x) = e^{x^2-1}$ at -1.

Solution: We have

$$h'(x) = e^{x^2-1}(2x)$$

so

$$h'(-1) = e^0(-2) = -2.$$

Further $h(-1) = e^0 = 1$.

Thus the equation of the tangent line is

$$y - 1 = -2(x+1)$$

or

$$y = -2x - 1.$$

(c) Use the Trapezoid rule and five intervals to estimate $\int_{-2}^{3} x^2 - x\,dx$.

Solution:

$$\begin{aligned}\int_{-2}^{3} x^2 - x\,dx &\approx \frac{6+2}{2} + \frac{2+0}{2} + \frac{0+0}{2} + \frac{2+0}{2} + \frac{6+2}{2} \\ &= \frac{1}{2}(8+2+0+2+8) = 10.\end{aligned}$$

Problem 3. Compute the following integrals:

(a) $\displaystyle\int 2x^3 \sin(x^2)\,dx =$

Solution:

We start with setting $u = x^2$ so $du = 2x\,dx$. Then

$$\begin{aligned}\int 2x^3 \sin(x^2)\,dx &= \int u \sin(u)\,du \\ &\text{Now we use integration by parts, setting} \\ &u = u, du = du, dv = \sin(u)\,du, v = -\cos u \\ &= -u\cos u + \int \cos u\,du \\ &= -u\cos u + \sin u + C \\ &= -x^2\cos(x^2) + \sin(x^2) + C.\end{aligned}$$

(b) $\displaystyle\int \frac{3x-3}{(x+1)(x-2)}\,dx =$

Solution:

We use a partial fractions decomposition.

$$\begin{aligned}\frac{3x-3}{(x+1)(x-2)} &= \frac{A}{x+1} + \frac{B}{x-2} \\ 3x - 3 &= A(x-2) + B(x+1)\end{aligned}$$

Plugging in $x = 2$ gives $3 = 3B$ so $B = 1$. Plugging in $x = -1$ gives $-6 = -3A$ so $A = 2$, and thus we have

$$\begin{aligned}\int \frac{3x-3}{(x+1)(x-2)}\,dx &= \int \frac{2}{x+1} + \frac{1}{x-2}\,dx \\ &= 2\ln|x+1| + \ln|x-2| + C.\end{aligned}$$

(c) $\displaystyle\int_0^1 \frac{1}{\sqrt{x^2+1}}\,dx =$

Solution:

We set $x = \tan\theta, dx = \sec^2\theta\, d\theta$, and we get a lower bound of 0 and an upper bound of $\pi/4$. Then

$$\begin{aligned}\int_0^1 \frac{1}{\sqrt{x^2+1}}\,dx &= \int_0^{\pi/4} \frac{1}{\sqrt{\tan^2\theta+1}} \sec^2\theta\, d\theta \\ &= \int_0^{\pi/4} \frac{1}{\sqrt{\sec^2\theta}} \sec^2\theta\, d\theta \\ &= \int_0^{\pi/4} \sec\theta\, d\theta \\ &= \ln|\sec\theta+\tan\theta|\Big|_0^{\pi/4} \\ &= \ln|\sqrt{2}+1| - \ln|1+0| = \ln|\sqrt{2}+1|.\end{aligned}$$

(Fun bonus fact: in class we briefly mentioned the function $\sinh(x) = \frac{e^x - e^{-x}}{2}$; this antiderivative we got is the inverse function $\operatorname{arcsinh}(x)$.)

Problem 4. (a) $\displaystyle\int_{-1}^1 \frac{dx}{\sqrt[3]{x}} =$

Solution: $\displaystyle\int_{-1}^1 \frac{dx}{\sqrt[3]{x}} = \int_{-1}^0 \frac{dx}{\sqrt[3]{x}} + \int_0^1 \frac{dx}{\sqrt[3]{x}}$. We have

$$\int_{-1}^0 \frac{dx}{\sqrt[3]{x}} = \lim_{t\to 0^-} \int_{-1}^t \frac{dx}{\sqrt[3]{x}} = \lim_{t\to 0^-} \frac{3}{2}x^{2/3}\Big|_{-1}^t = \lim_{t\to 0^-} \frac{3}{2}\left(\sqrt[3]{t^2} - (-1)^2\right) = \frac{-3}{2}.$$

$$\int_0^1 \frac{dx}{\sqrt[3]{x}} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{\sqrt[3]{x}} = \lim_{t\to 0^+} \frac{3}{2}x^{2/3}\Big|_t^1 = \lim_{t\to 0^+} \frac{3}{2}\left(1 - \sqrt[3]{t^2}\right) = \frac{3}{2}.$$

Thus our original integral is $\displaystyle\int_{-1}^1 \frac{dx}{\sqrt[3]{x}} = \frac{-3}{2} + \frac{3}{2} = 0.$

(b) Set up (but do not compute!) an integral for the arc length of the curve $y = x^2 + 2x$ between $x = 0$ and $x = 2$.

Solution: We have $y' = 2x + 2$ and so $ds = \sqrt{1 + (2x+2)^2}\,dx$. Then the arc length is

$$L = \int_0^2 \sqrt{1+(2x+2)^2}\,dx = \int_0^2 \sqrt{5+8x+4x^2}\,dx$$

(c) What is the surface area of the curve $y = x^2 + 2$ between $x = 0$ and $x = 2$ rotated around the y-axis?

Solution:

The obvious approach is maybe a little hard. We can take $x = \sqrt{y-2}$, so that $\frac{dx}{dy} = \frac{1}{2\sqrt{y-2}}$. As x goes from 0 to 2 we have y going from 2 to 6, and we get

$$\begin{aligned}\int_2^6 2\pi\sqrt{y-2}\sqrt{1+\frac{1}{4(y-2)}}\,dy &= 2\pi\int_2^6 \sqrt{y-2}\sqrt{\frac{4y-7}{4y-8}}\,dy \\ &= 2\pi\int_2^6 \frac{1}{2}\sqrt{4y-7}\,dy \\ &= \pi\int_2^6 \sqrt{4y-7}\,dy.\end{aligned}$$

At this point we can set $u = 4y - 7$ so $du = 4\,dy$, and get

$$\begin{aligned}
\pi \int_2^6 \sqrt{4y-7}\,dy &= \pi \int_1^{17} \frac{1}{4}\sqrt{u}\,du \\
&= \frac{\pi}{4} \cdot \frac{2}{3} u^{3/2}\Big|_1^{17} \\
&= \frac{\pi}{6}\left(17\sqrt{17} - 1\right) \approx 11.5\pi \approx 36.18.
\end{aligned}$$

But there's a probably slightly simpler way. We can do everything in terms of x: we have $y' = 2x$, so $ds = \sqrt{1 + (2x)^2}\,dx$.

$$\begin{aligned}
A &= \int_0^2 2\pi x \sqrt{1 + (2x)^2}\,dx = 2\pi \int_0^2 x\sqrt{1 + 4x^2}\,dx \\
&\quad u = 1 + 4x^2, du = 8x\,dx \\
&= 2\pi \int_1^{17} \sqrt{u}\frac{dx}{8} = 2\pi \frac{1}{8} \cdot \frac{2}{3} u^{3/2}\Big|_1^{17} = \frac{\pi}{6}\left(17\sqrt{17} - 1\right) \approx 11.5\pi \approx 36.18.
\end{aligned}$$