Math 1232 Practice Final Solutions

Instructor: Jay Daigle

1. This test is due at the scheduled exam time. Logistically, this will work just like the mastery quizzes: download it, write up your answers, and upload them to Blackboard for us to grade.
2. You will have three hours for this test. Please write down your start and end times on the test and include that in your upload. You may not spend more than two hours on the test unless you have a specific accommodation.
3. You are not allowed to consult books or notes during the test, but you may use a two-page cheat sheet you have made for yourself ahead of time. Please upload your sheet along with your test.
4. If you have questions, I will be online and responsive during the scheduled exam time. If you want to take the test at a time you know I'll be able to answer any questions quickly, I encourage you to use that time slot.
5. You may use a calculator, but don't use a graphing calculator or anything else that can do symbolic computations. Using a calculator for basic arithmetic is fine.

Name:

Time Started:

Time Completed:

1	
2	
3	
4	
5	
Σ	

Problem 1.

(a) $\int \sin x \cos 2x \, dx$

Solution: Take $u = \cos 2x$ and $dv = \sin x \, dx$. We get $du = -2 \sin 2x \, dx$ and $v = -\cos x \, dx$, and

$$\begin{aligned}
\int \sin x \cos 2x \, dx &= -\cos 2x \cos x - \int 2 \sin 2x \cos x \, dx \\
&= -\cos 2x \cos x - 2\left(\int \sin 2x \cos x \, dx\right) \\
&= -\cos 2x \cos x - 2\left(\sin x \sin 2x - 2\int \sin x \cos 2x \, dx\right) \\
&= -\cos x \cos 2x - 2 \sin x \sin 2x + 4 \int \sin x \cos 2x \, dx \\
-3 \int \sin x \cos 2x \, dx &= -\cos x \cos 2x - 2 \sin x \sin 2x \\
\int \sin x \cos 2x \, dx &= \frac{1}{3}(\cos x \cos 2x + 2 \sin x \sin 2x)
\end{aligned}$$

(b) $\int_0^\pi \sin^4(x) \, dx$

Solution:

$$\begin{aligned}
\int_0^\pi \sin^4(x) \, dx &= \int_0^\pi \left(\frac{1 - \cos 2x}{2}\right)^2 dx = \frac{1}{4}\int_0^\pi 1 - 2\cos 2x + \cos^2 2x \, dx \\
&= \frac{1}{4}\int_0^\pi 1 - 2\cos 2x + \frac{1 + \cos 4x}{2} \, dx = \frac{1}{4}\left(x - \sin 2x + \frac{x}{2} + \frac{\sin 4x}{8}\right)\Big|_0^\pi \\
&= \frac{1}{4}\left(\pi - 0 + \frac{\pi}{2} + \frac{0}{8} - 0 - 0 - 0 - 0\right) = \frac{3\pi}{8}.
\end{aligned}$$

(c) $\int_{\sqrt{7}}^{2\sqrt{7}} \frac{dx}{x\sqrt{x^2 - 7}}$

Solution: We see as $\sqrt{x^2 - 7}$, which should make us think of trigonometric substitution, and in particular $\sqrt{7} \sec\theta = x$. (In the original version of the practice final I posted I had a typo here; see below). We work out $dx = \sqrt{7} \sec\theta \tan\theta \, d\theta$, and the bounds now range from $\sec\theta = 1$ to $\sec\theta = 2$, and thus $\theta = 0$ to $\theta = \pi/3$. Thus

$$\begin{aligned}
\int_{\sqrt{7}}^{2\sqrt{7}} \frac{dx}{x\sqrt{x^2 - 7}} &= \int_0^{\pi/3} \frac{\sqrt{7}\sec\theta \tan\theta \, d\theta}{\sqrt{7}\sec\theta\sqrt{7\sec^2\theta - 7}} \\
&= \int_0^{\pi/3} \frac{\sec\theta \tan\theta \, d\theta}{\sec\theta\sqrt{7\tan^2\theta}} \\
&= \int_0^{\pi/3} \frac{d\theta}{\sqrt{7}} = \frac{\theta}{\sqrt{7}}\Big|_0^{\pi/3} = \frac{\pi}{3\sqrt{7}}.
\end{aligned}$$

(d) $\int_1^{+\infty} \frac{1}{x^2 - 2x} \, dx$

Solution:

$$\begin{aligned}
\int_1^{+\infty} \frac{1}{x^2-2x}\,dx &= \lim_{t\to+\infty}\int_1^t \frac{dx}{x^2-2x} \\
&= \lim_{t\to+\infty}\int_1^2 \frac{dx}{x(x-2)} + \int_2^t \frac{dx}{x(x-2)} \\
&= \lim_{r\to 2^-}\int_1^r \frac{dx}{x(x-2)} + \lim_{s\to 2^+}\int_s^3 \frac{dx}{x(x-2)} + \lim_{t\to+\infty}\int_3^t \frac{dx}{x(x-2)}
\end{aligned}$$

The integral converges if and only if each of these three integrals converges. But let's consider the first one:

$$\begin{aligned}
\lim_{r\to 2^-}\int_1^r \frac{dx}{x(x-2)} &= \lim_{r\to 2^-}\frac{1}{2}\int_1^r \frac{1}{x-2} - \frac{1}{x}\,dx \\
&= \frac{1}{2}\lim_{r\to 2^-} \left(\ln|x-2| - \ln|x|\right)\Big|_1^r \\
&= \frac{1}{2}\lim_{r\to 2^-} \left(\ln(2-r) - \ln(r) - \ln(1) - \ln(1)\right) \\
&= \frac{1}{2}\lim_{r\to 2^-} \left(\ln(2-r) - \ln(r)\right) = -\infty.
\end{aligned}$$

So one of the summands doesn't converge, and thus the integral as a whole diverges.

Problem 2.

(a) Analyze the convergence of $\displaystyle\sum_{n=2}^{\infty}\frac{3(-1)^n}{n\ln(n)}$. **Solution:** The terms of this series are decreasing and tend to zero, and the series is clearly alternating, so by the alternating series test the series converges. Once we take the absolute value, we have the series $\sum \frac{3}{n\ln n}$. We can't compare this to $\frac{1}{n}$ because that's unhelpful. You may remember from class that this diverges; if not, we compute

$$\begin{aligned}
\int_1^{+}\infty \frac{1}{x\ln x}\,dx &= \lim_{t\to+\infty}\int_1^t \frac{1}{x\ln x}\,dx \\
&= \lim_{t\to+\infty}\int_1^{\ln t} \frac{1}{u}\,du \\
&= \lim_{t\to+\infty} \ln u\Big|_1^{\ln t} = \lim_{t\to+\infty} \ln\ln t - \ln 1 = +\infty
\end{aligned}$$

so by the integral test the series does not converge absolutely. Thus the series converges conditionally.

(b) Analyze the convergence of $\displaystyle\sum_{n=1}^{\infty}(-1)^n\left(\frac{5n+7}{8n-4}\right)^n$.

Solution: We use the root test. We have

$$\lim_{n\to+\infty}\left|(-1)^n\frac{5n+7}{8n-4}\right| = \frac{5}{8} < 1$$

so the series converges absolutely.

(c) Analyze the convergence of $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{n^3+n^2+n+1}{\sqrt{n^9}}$.

Solution: We consider the absolute value of this sequence. The sequence

$$\sum_{n=1}^{\infty}\left|(-1)^n\frac{n^3+n^2+n+1}{\sqrt{n^9}}\right| = \sum_{n=1}^{\infty}\frac{n^3+n^2+n+1}{\sqrt{n^9}}$$

is positive, so we can use the Limit Comparison Test. We have

$$\lim_{n\to\infty} \frac{(n^3+n^2+n+1)/\sqrt{n^9}}{1/\sqrt{n^3}} = \lim_{n\to\infty} \frac{n^{9/2}+n^{7/2}+n^{5/2}+n^{3/2}}{n^{9/2}} = \lim_{n\to\infty} \frac{1+1/n+1/n^2+1/n^3}{1} = 1.$$

Thus by the Limit Comparison Test, our series converges if and only if $\sum \frac{1}{n^{3/2}}$ converges. But $3/2 > 1$ so this converges, and thus our series converges (absolutely).

Problem 3.

(a) Find the radius and interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{(x-3)^n}{(2n)^2+1}$.

Solution: We use the ratio test to find the radius of convergence. We have

$$\lim_{n\to\infty} \left| \frac{(x-3)^{n+1}/((2n+1)^2+1)}{(x-3)^n/((2n)^2+1)} \right| = \lim_{n\to\infty} \frac{|(x-3)|(4n^2+1)}{4n^2+4n+2} = |x-3|.$$

Thus the series converges absolutely when $|x-3| < 1$ and diverges when $|x-3| > 1$, and thus it converges absolutely on $(2,4)$.

When $|x-3| = 1$ we have two points to check. If $x = 4$ then our series is $\sum \frac{1}{(2n)^2+1}$ which converges by the comparison test, since $\frac{1}{(2n)^2+1} < \frac{1}{n^2}$. If $x = 2$ then our series is $\sum \frac{(-1)^n}{(2n)^2+1}$ which converges by the alternating series test. Thus the real interval of convergence is $[2,4]$.

(b) Find a power series for $x^2 \arctan(x^2)$ centered at 0.

Solution: We know that $\arctan(x) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}$. Thus

$$x^2 \arctan(x^2) = x^2 \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+4}}{2n+1}.$$

(c) Find the Taylor series for $f(x) = \frac{3}{x^3}$ centered at 3.

Solution: First we compute some derivatives:

$$\begin{aligned} f(x) &= \frac{3}{x^3} & f(3) &= \frac{1}{9} \\ f'(x) &= -3\frac{3}{x^4} & f'(3) &= -\frac{1}{9} \\ f''(x) &= (3)(4)\frac{3}{x^5} & f''(3) &= \frac{4}{27} \\ f'''(x) &= -(3)(4)(5)\frac{3}{x^6} & f'''(3) &= -\frac{20}{81} \end{aligned}$$

By now we can see the pattern: $f^{(n)}(x) = (-1)^n \frac{(n+2)!}{2} \frac{3}{x^{3+n}}$ and thus $f^{(n)}(3) = (-1)^n \frac{(n+2)!}{2 \cdot 3^{2+n}}$. So the Taylor series is

$$T_f(x,3) = \sum_{n=0}^{\infty} (-1)^n \frac{(n+2)!}{n! \cdot 2 \cdot 3^{2+n}} (x-3)^n = \sum_{n=0}^{\infty} (-1)^n \frac{(n+2)(n+1)}{2 \cdot 3^{2+n}} (x-3)^n$$

Problem 4.

(a) Find the slope of the tangent line to the curve defined by the polar equation $r = 2 + \sin(3\theta)$ at the point $\theta = \pi/4$.

Solution:

We have

$$
\begin{aligned}
x &= 2\cos(\theta) + \cos(\theta)\sin(3\theta) \\
y &= 2\sin(\theta) + \sin(\theta)\sin(3\theta) \\
x(\pi/4) &= \sqrt{2} + 1/2 \\
y(\pi/4) &= \sqrt{2} + 1/2 \\
\frac{dx}{d\theta} &= -2\sin(\theta) - \sin(\theta)\sin(3\theta) + 3\cos(\theta)\cos(3\theta) \\
\frac{dy}{d\theta} &= 2\cos(\theta) + \cos(\theta)\sin(3\theta) + 3\sin(\theta)\cos(3\theta) \\
\frac{dy}{dx} &= \frac{dy/d\theta}{dx/d\theta} = \frac{2\cos(\theta) + \cos(\theta)\sin(3\theta) + 3\sin(\theta)\cos(3\theta)}{-2\sin(\theta) - \sin(\theta)\sin(3\theta) + 3\cos(\theta)\cos(3\theta)} \\
&= \frac{\sqrt{2} + 1/2 - 3/2}{-\sqrt{2} - 1/2 - 3/2} = \frac{\sqrt{2} - 1}{-2 - \sqrt{2}} = -1 \\
y - (\sqrt{2} + 1/2) &= \frac{1 - \sqrt{2}}{2 + \sqrt{2}}(x - (\sqrt{2} + 1/2)).
\end{aligned}
$$

(b) Find the solution to $y' = x^2 y^3$ if $y(0) = 1$.

Solution:

$$
\begin{aligned}
\frac{dy}{dx} &= x^2 y^3 \\
\frac{dy}{y^3} &= x^2\,dx \\
\int \frac{dy}{y^3} &= \int x^2\,dx \\
\frac{-1}{2y^2} &= \frac{x^3}{3} + C \\
y^2 &= \frac{-1}{2x^3/3 + 2C}
\end{aligned}
$$

Plugging in $x = 0, y = 1$ gives

$$
\begin{aligned}
1 &= \frac{-1}{2C} \\
C &= -1/2 \\
y &= \sqrt{\frac{1}{1/2 - 2x^3/3}}.
\end{aligned}
$$

(c) Find the area of the surface obtained by rotating the curve $x = 1 + 2y^2$ for $1 \le y \le 2$ about the x-axis.

Solution: Recall we have the formula for surface area $A = \int 2\pi y\,ds$ when we rotate around the x-axis. We will further integrate with respect to y because everything is given as a function of y. We get $y' = 4y$, and thus $ds = \sqrt{1 + 16y^2}$, so

$$
\begin{aligned}
SA &= \int_1^2 2\pi y\sqrt{1 + 16y^2}\,dy \\
&\quad u = 1 + 16y^2, du = 32y\,dy \\
&= \int_{17}^{65} \frac{\pi}{16}\sqrt{u}\,du \\
&= \frac{\pi}{16}\frac{2u^{3/2}}{3}\Big|_{17}^{65} = \frac{\pi}{24}\left(65\sqrt{65} - 17\sqrt{17}\right).
\end{aligned}
$$

Problem 5.

(a) Let $g(x) = \sqrt[5]{x^9 + x^7 + x + 1}$. Find $(g^{-1})'(1)$.

Solution: We see that $g(0) = 1$, so $g^{-1}(1) = 0$. Then by the Inverse Function Theorem we have

$$\begin{aligned}(g^{-1})'(1) &= \frac{1}{g'(g^{-1}(1))} = \frac{1}{g'(0)} \\ g'(x) &= \frac{1}{5}(x^9 + x^7 + x + 1)^{-4/5}(9x^8 + 7x^6 + 1) \\ g'(0) &= \frac{1}{5}(1)(1) = \frac{1}{5}\end{aligned}$$

$(g^{-1})'(1) = 5.$

(b) Approximate $\displaystyle\int_1^5 3^x\,dx$ with four intervals and Simpson's Rule.

Solution:

$$\begin{aligned}\int_1^5 f(x)\,dx &\approx \frac{1}{3}\left(f(1) + 4f(2) + 2f(3) + 4f(4) + f(5)\right) \\ \int_1^5 3^x\,dx &\approx \frac{1}{3}\left(3^1 + 4\cdot 3^2 + 2\cdot 3^3 + 4\cdot 3^4 + 3^5\right) \\ &= 1 + 12 + 18 + 108 + 81 = 220.\end{aligned}$$

(c) Use a second-degree Taylor polynomial to approximate $\sqrt[4]{82}$.

Solution: If $g(x) = \sqrt[4]{1+x}$, then by the binomial series we have $g(x) \approx 1 + \frac{x}{4} - \frac{3x^2}{32}$. Then

$$\begin{aligned}\sqrt[4]{82} = \sqrt[4]{81+1} = 3\sqrt[4]{1 + 1/81} &\approx 3\left(1 + \frac{1}{81\cdot 4} - \frac{3}{32\cdot 81^2}\right) \\ &= 3 + \frac{1}{27\cdot 4} - \frac{1}{32\cdot 27^2} \\ &= 3 + \frac{1}{108} - \frac{1}{23328} = \frac{70119}{23328} \approx 3.00579.\end{aligned}$$