Problem 1. (a) Let $f(x) = \sqrt[3]{x^5 + x^4 + x^3 + x^2 + 2x}$. Find $(f^{-1})'(4)$.

Solution: Plugging in numbers, we see that $f(2) = \sqrt[3]{32 + 16 + 8 + 4 + 4} = \sqrt[3]{64} = 4$. Then by the Inverse Function Theorem we have $(f^{-1})'(4) = \frac{1}{f'(2)}$. But

$$f'(x) = \frac{1}{3}\left(x^5 + x^4 + x^3 + x^2 + 2x\right)^{-2/3}\left(5x^4 + 4x^3 + 3x^2 + 2x + 2\right)$$
$$f'(2) = \frac{1}{3}(64)^{-2/3}(80 + 32 + 12 + 4 + 2) = \frac{130}{48} = \frac{65}{24}.$$

Thus by the inverse function theorem we have

$$(f^{-1})'(4) = \frac{24}{65}.$$

(b) Find $\lim_{x\to 0} \frac{2\sin(x) - \sin(2x)}{x - \sin(x)}$.

Solution: $\lim_{x\to 0} 2\sin(x) - \sin(2x) = 0 - 0 = 0$, and $\lim_{x\to 0} x - \sin(x) = 0$, so we can use L'Hôpital's Rule.

$$\begin{aligned}\lim_{x\to 0} \frac{2\sin(x) - \sin(2x)}{x - \sin(x)} &= \lim_{x\to 0} \frac{2\cos(x) - 2\cos(2x)}{1 - \cos(x)} \to \frac{0}{0} \\ &= \lim_{x\to 0} \frac{-2\sin(x) + 4\sin(2x)}{\sin(x)} \to \frac{0}{0} \\ &= \lim_{x\to 0} \frac{-2\cos(x) + 8\cos(2x)}{\cos(x)} = \frac{6}{1} = 6.\end{aligned}$$

(c) Compute the following. In all cases your answers should be exact, with no decimals, and no logs or exponentials or trig functions..

$$\ln(e^3) + \ln(3) + \ln(e/3) =$$

Solution: $3 + \ln(3 \cdot e/3) = 3 + 1 = 4$

$$\arcsin(-\sqrt{2}/2) =$$

Solution: $-\pi/4$

$$\cos(\arcsin(3/7) =$$

Solution: $\frac{\sqrt{49-9}}{7} = \frac{2\sqrt{10}}{7}$

Problem 2. (a) Compute $g'(4)$ where $g(x) = \ln(x^3 + 3x + \sqrt{x})$.

Solution:

$$g'(x) = \frac{1}{x^3 + 3x + \sqrt{x}}\left(3x^2 + 3 + \frac{1}{2\sqrt{x}}\right)$$

so

$$g'(4) = \frac{1}{4^3 + 3\cdot 4 + \sqrt{4}}\left(3(4^2) + 3 + \frac{1}{2\sqrt{4}}\right) = \frac{51 + \frac{1}{4}}{78} = \frac{205}{312}.$$

(b) Find the tangent line to $h(x) = \arcsin(e^x)$ at $\ln(1/2)$.

Solution: We have $h'(x) = \frac{1}{\sqrt{1-e^{2x}}} \cdot e^x$, so $h'(\ln(1/2)) = \frac{e^{\ln 1/2}}{\sqrt{1-e^{2\ln(1/2)}}} = \frac{1/2}{\sqrt{1-1/4}} = \frac{1}{\sqrt{3}}$. We also have $h(\ln(1/2)) = \arcsin(1/2) = \pi/6$.

Thus the equation of the tangent line is

$$y - \pi/6 = \frac{1}{\sqrt{3}}(x - \ln(1/2)).$$

(c) Use Simpson's rule and six intervals to estimate $\int_0^6 x^4\,dx$.

Solution:

$$\begin{aligned}\int_0^6 x^4\,dx &\approx \frac{1}{3}\left(0^4 + 4\cdot 1^4 + 2\cdot 2^4 + 4\cdot 3^4 + 2\cdot 4^4 + 4\cdot 5^4 + 6^4\right)\\ &= \frac{1}{3}\left(0 + 4 + 32 + 324 + 512 + 2500 + 1296\right) = \frac{4668}{3} = 1556.\end{aligned}$$

Problem 3. Compute the following integrals:

(a) $\displaystyle\int \frac{2x+1}{\sqrt{x^2-1}}\,dx$

Solution: Since we see $\sqrt{x^2-1}$ we want to try a trig substitution. (You might try $u = x^2 - 1$ first, which almost works, but doesn't quite). So we set $x = \sec\theta$ and $dx = \sec\theta\tan\theta\,d\theta$. We have

$$\begin{aligned}\int \frac{2x+1}{\sqrt{x^2-1}}\,dx &= \int \frac{2\sec\theta+1}{\sqrt{\sec^2\theta-1}}\sec\theta\tan\theta\,d\theta\\ &= \int \frac{2\sec^2\theta\tan\theta + \sec\theta\tan\theta}{\tan\theta}\,d\theta\\ &= \int 2\sec^2\theta + \sec\theta\,d\theta\\ &= 2\tan\theta + \ln|\sec\theta + \tan\theta| + C.\end{aligned}$$

If $\sec\theta = x$ then θ is in a triangle with hypotenuse x and adjacent side 1 and thus opposite side $\sqrt{x^2-1}$. Thus $\tan\theta = \sqrt{x^2-1}$. This is good, since this formula appeared in our original question, and we see that

$$\int \frac{2x+1}{\sqrt{x^2-1}}\,dx = 2\sqrt{x^2-1} + \ln\left|x + \sqrt{x^2-1}\right| + C.$$

(b) $\displaystyle\int x\sec^2 x\,dx$

Solution: We use integration by parts. Take $u = x, dv = \sec^2 x\,dx$ so $du = dx, v = \tan x$. Then

$$\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx = x\tan x + \ln|\cos x| + C.$$

(c) $\displaystyle\int_0^1 \frac{3x^2-6x+1}{(x^2-x-1)(x-2)}\,dx$

Solution: We use a partial fractions decomposition.

$$\begin{aligned}\frac{3x^2-6x+1}{(x^2-x-1)(x-2)} &= \frac{A}{x-2} + \frac{Bx+C}{x^2-x-1}\\ 3x^2-6x+1 &= A(x^2-x-1) + (Bx+C)(x-2).\end{aligned}$$

Plugging in $x = 2$ gives us that $1 = A$. Plugging in $x = 0$ gives $1 = -A - 2C = -1 - 2C$ and thus $C = -1$. Then plugging in $x = 1$ gives $-2 = -A - B - C = -1 - B + 1$ and thus $B = 2$. So we have

$$\begin{aligned}\int_0^1 \frac{3x^2 - 6x + 1}{(x^2 - x - 1)(x - 2)}\,dx &= \int_0^1 \frac{1}{x-2} + \frac{2x-1}{x^2 - x - 1}\,dx \\ &= \left(\ln|x-2| + \ln|x^2 - x - 1|\right)\Big|_0^1 \\ &= \ln(1) + \ln(1) - \ln(2) - \ln(1) = -\ln(2).\end{aligned}$$

Problem 4. (a) Does $\displaystyle\int_0^\infty \frac{x}{x^3+1}\,dx$ converge or diverge? Why?

Solution:

We split the integral up into two parts:

$$\int_0^\infty \frac{x}{x^3+1}\,dx = \int_0^1 \frac{x}{x^3+1}\,dx + \int_1^\infty \frac{x}{x^3+1}\,dx.$$

The first integral is a finite integral of a continuous function and thus converges. Then for $x \geq 1$ we have $\frac{x}{x^3+1} \leq \frac{x}{x^3} = \frac{1}{x^2}$. From class we know that $\int_1^{+\infty} \frac{1}{x^2}\,dx$ converges, so $\int_1^{+\infty} \frac{x}{x^3+1}\,dx$ also converges. Thus the original integral is convergent.

(b) Find the arc length of the curve $y = \frac{1}{3}(2 + x^2)^{3/2}$ from between $x = 0$ and $x = 2$.

Solution: We have $y' = \frac{1}{2}\sqrt{2 + x^2} \cdot 2x = x\sqrt{2 + x^2}$, so $ds = \sqrt{1 + x^2(2 + x^2)}dx = \sqrt{1 + 2x^2 + x^4}dx = (x^2 + 1)dx$. So the arc length is

$$L = \int_0^2 (x^2 + 1)\,dx = \frac{x^3}{3} + x\Big|_0^2 = \frac{8}{3} + 2 = \frac{14}{3}.$$

(c) Find the surface area of the surface obtained by rotating $y = \sqrt{5 + 4x}$ for $-1 \leq x \leq 1$ about the x-axis.

Solution: We have $y' = \frac{1}{2}(5 + 4x)^{-1/2} \cdot 4 = \frac{2}{\sqrt{5+4x}}$, so $ds = \sqrt{1 + \frac{4}{5+4x}}dx$. Then

$$\begin{aligned}A &= \int_{-1}^1 2\pi y\,ds = 2\pi \int_{-1}^1 \sqrt{5+4x}\sqrt{1 + \frac{4}{5+4x}}dx \\ &= 2\pi \int_{-1}^1 \sqrt{5 + 4x + 4}\,dx = 2\pi \int_{-1}^1 \sqrt{9 + 4x}\,dx \\ &= 2\pi \left(\frac{2}{3}(9 + 4x)^{3/2} \cdot \frac{1}{4}\right)\Bigg|_{-1}^1 = 2\pi\left(\frac{1}{6}13\sqrt{13} - \frac{1}{6}5\sqrt{5}\right) = \frac{\pi}{3}\left(13\sqrt{13} - 5\sqrt{5}\right).\end{aligned}$$