3 Applications of the Integral

Now that we've learned some techniques for doing more complicated integrals, we want to use them to accomplish something. Why do we want to compute integrals, and what can they do for us?

3.1 Improper Integrals and Unbounded Area

Recall that integrals were originally defined to compute areas. But so far we've only looked at the areas of regions that are, in some sense, "bounded": we may need calculus to find the exact area of the region, but we know the area is finite (and thus is a number)because we can draw a big circle around the whole shape. But sometimes we want to find the area of shapes that extend infinitely in one direction.

Example 3.1 (Motivating Example)**.** What is the area of the region bounded by the x-axis, the line $x = 1$, and the curve $y = 1/x^2$? Notice this region doesn't have any boundary at all on the right edge.

At first we don't know what to do, since our integrals are only defined on finite intervals. But we imagine the "remaining" area of the region must get smaller and smaller as x gets bigger and bigger. So what happens if we take the integral of a big chunk of the region?

$$A_N = \int_1^N \frac{1}{x^2}\,dx = \frac{-1}{x}\Big|_1^N = \frac{-1}{N} - \frac{-1}{1} = 1 - \frac{1}{N}.$$

As N gets very large, we see that this area approaches 1, so we conclude the area of the whole region is 1.

There are two different ways for regions to be unbounded; it's entirely possible for both to happen at once, but we can always separate them and deal with them separately. We call such integrals *improper integrals.*

3.1.1 Improper integrals to ∞

The first situation is the situation in our motivating example, where we have to integrate over an "infinitely wide region."

Definition 3.2. If $\int_a^t f(x)\,dx$ exists for every $t \geq a$, then we define the *improper integral*

$$\int_a^{+\infty} f(x)\,dx = \lim_{t\to+\infty} \int_a^t f(x)\,dx$$

provided this limit exists.

If $\int_t^b f(x)\,dx$ exists for every $t \leq b$, we define

$$\int_{-\infty}^{b} f(x)\,dx = \lim_{t\to-\infty}\int_t^b f(x)\,dx$$

provided the limit exists.

We say these integrals are *convergent* if the limit exists and *divergent* if the limit does not exist; and if both integrals are convergent, we write

$$\int_{-\infty}^{+\infty} f(x)\,dx = \int_{-\infty}^{a} f(x)\,dx + \int_a^{+\infty} f(x)\,dx.$$

In this case it doesn't matter which a we use.

Remark 3.3. We can recast the previous example as showing that

$$\int_1^{+\infty} \frac{1}{x^2}\,dx = 1.$$

Example 3.4. What is $\int_1^{+\infty} \frac{1}{x}\,dx$?

$$\begin{aligned}\int_1^{+\infty} \frac{1}{x}\,dx &= \lim_{t\to+\infty}\int_1^t \frac{1}{x}\,dx = \lim_{t\to+\infty}\left(\ln|x|\right)\Big|_1^t \\ &= \lim_{t\to+\infty}\left(\ln|t| - \ln|1|\right) = \lim_{t\to+\infty}\ln|t| = +\infty.\end{aligned}$$

Thus this integral is divergent. Geometrically, this means that the area under this curve is in fact infinite.

Remark 3.5. It turns out that $\int_1^{+\infty} x^r\,dx$ is convergent whenever $r < -1$ and divergent whenever $r \geq -1$. This is worked out in your textbook, and we'll return to it later.

Example 3.6. What is $\int_{-\infty}^{2} \sin(x)\,dx$?

We write this as a limit:

$$\begin{aligned}\int_{-\infty}^{2} \sin(x)\,dx &= \lim_{t\to-\infty}\int_t^2 \sin(x)\,dx \\ &= \lim_{t\to-\infty}\left(-\cos(x)\right)\Big|_t^2 \\ &= \lim_{t\to-\infty}\left(-\cos(2) - (-\cos(t))\right) = \lim_{t\to-\infty}\cos(t) - \cos(2).\end{aligned}$$

This limit does not exist (because cos is periodic), so the integral is divergent. Note that in this case the (net) area isn't infinite; it just isn't well-defined.

Example 3.7. What is $\int_{-\infty}^{+\infty} \frac{1}{1+x^2}\,dx$?

First think about what you expect to happen. The graph of this function peaks in the middle at $(0,1)$, and trails off to zero as x gets large or small. So it's plausible that this integral is finite. It is certainly positive.

We can pick any a we want, and it's convenient to pick $a = 0$ so things are symmetrical.

$$\begin{aligned}
\int_{-\infty}^{+\infty} \frac{dx}{1+x^2} &= \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{+\infty} \frac{dx}{1+x^2} \\
&= \lim_{t\to-\infty} \int_t^0 \frac{dx}{1+x^2} + \lim_{s\to+\infty} \int_0^s \frac{dx}{1+x^2} \\
&= \lim_{t\to-\infty} \arctan(x)|_t^0 + \lim_{s\to+\infty} \arctan|_0^s \\
&= \lim_{t\to-\infty} (\arctan(0) - \arctan(t)) + \lim_{s\to+\infty} (\arctan(s) - \arctan(0)) \\
&= -\lim_{t\to-\infty} \arctan(t) + \lim_{s\to+\infty} \arctan(s) = -(-\pi/2) + \pi/2 = \pi.
\end{aligned}$$

Both partial integrals are convergent, so the total integral is convergent and the area under the curve is π.

Example 3.8. $\int_{-\infty}^{+\infty} 2xe^{-x^2}\,dx$.

We again split this into two integrals. We will also set $u = x^2, du = 2x\,dx$.

$$\begin{aligned}
\int_{-\infty}^{+\infty} 2xe^{-x^2}\,dx &= \lim_{t\to-\infty} \int_t^0 2xe^{-x^2}\,dx + \lim_{s\to+\infty} \int_0^s 2xe^{-x^2}\,dx \\
&= \lim_{t\to-\infty} \int_{t^2}^0 e^{-u}\,du + \lim_{s\to+\infty} \int_0^{s^2} e^{-u}\,du \\
&= \lim_{t\to-\infty} -e^{-u}|_{t^2}^0 + \lim_{s\to+\infty} -e^{-u}|_0^{s^2} \\
&= \lim_{t\to-\infty} -e^0 - (-e^{-t^2}) + \lim_{s\to+\infty} -e^{-s^2} - (-e^0) = -1 - 0 + 0 + 1 = 0.
\end{aligned}$$

3.1.2 Improper integrals of discontinuous functions

There's a completely separate type of problem, where our x-values are bounded but our function behaves badly somewhere in that interval. Generally the issue comes up when our region is infinite in the *vertical* direction.

Example 3.9 (Motivating Example)**.** What is the area under $f(x) = 1/\sqrt{x}$ between $x = 0$ and $x = 1$?

f isn't well-defined at 0, so we can't just use our normal integral. But we can find the area of almost all of the region; if ε is a small number, we have

$$\int_\varepsilon^1 x^{-1/2}\,dx = 2x^{1/2}|_\varepsilon^1 = 2(1-\sqrt{\varepsilon}).$$

It's easy to calculuate that $\lim_{\varepsilon\to 0} 2(1-\sqrt{\varepsilon}) = 2$, so we say the area of this region is 2.

Definition 3.10. If f is continuous on $[a, b)$ but discontinuous at b, we define the improper integral

$$\int_a^b f(x)\,dx = \lim_{t\to b^-} \int_a^t f(x)\,dx$$

provided the limit exists (and is finite).

If f is continuous on $(a, b]$ but discontinuous at a, we define

$$\int_a^b f(x)\,x = \lim_{t\to a^+} \int_t^b f(x)\,dx$$

provided the limit exists.

Again, the improper integral $\int_a^b f(x)\,dx$ is convergent if the limit exists, and divergent if it does not.

If f has a discontinuity at c for $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

Example 3.11. Our previous example showed that $\int_0^1 x^{-1/2}\,dx = 2$.

Example 3.12. What is $\int_0^{\pi/2} \tan x\,dx$?

$\tan(\pi/2)$ is not well defined. So we write

$$\begin{aligned}\int_0^{\pi/2} \tan x\,dx &= \lim_{t\to\pi/2^-} \int_0^t \tan x\,dx \\ &= \lim_{t\to\pi/2^-} \ln|\sec x||_0^t \\ &= \lim_{t\to\pi/2^-} \ln|\sec t| - \ln|1| = +\infty\end{aligned}$$

since $\lim_{t\to\pi/2^-} \sec t = +\infty$. So the integral is divergent.

Example 3.13 (Warning Example)**.** What is $\int_0^{\pi} \tan x\,dx$?

If we're sloppy, we might reason as follows:

$$\int_0^{\pi} \tan x\,dx = \ln|\sec x||_0^{\pi} = \ln|-1| - \ln|1| = 0.$$

This is false because there is a discontinuity in the middle. We would need to split this integral into

$$\int_0^{\pi/2} \tan x\,dx + \int_{\pi/2}^{\pi} \tan x\,dx$$

and we already saw that the first integral is divergent (as is the second), so the whole integral is also divergent.

Example 3.14. What is $\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx$?

This is an improper integral since $\frac{1}{\sqrt[3]{x-1}}$ isn't defined at 1. We break it apart and compute a limit:

$$\begin{aligned}
\int_0^2 \frac{1}{\sqrt[3]{x-1}}\,dx &= \int_0^1 \frac{1}{\sqrt[3]{x-1}}\,dx + \int_1^2 \frac{1}{\sqrt[3]{x-1}}\,dx \\
&= \lim_{t\to 1^-} \int_0^t \frac{1}{\sqrt[3]{x-1}}\,dx + \lim_{s\to 1^+} \int_s^2 \frac{1}{\sqrt[3]{x-1}}\,dx \\
&= \lim_{t\to 1^-} (3/2(x-1)^{2/3})|_0^t - \lim_{s\to 1^+} (3/2(x-1)^{2/3})|_s^2 \\
&= \lim_{t\to 1^-} (3/2(t-1)^{2/3} - 3/2\cdot 1) + \lim_{s\to 1^+} (3/2\cdot 1 - 3/2(s-1)^{2/3}) \\
&= 0 - 3/2 + 3/2 - 0 = 0.
\end{aligned}$$

Example 3.15. $\int_0^{+\infty} \frac{1}{(x-6)^2}\,dx$.

We have to split this in two places, since our function is discontinuous at 6. We have

$$\begin{aligned}
\int_0^{+\infty} \frac{1}{(x-6)^2}\,dx &= \int_0^6 \frac{dx}{(x-6)^2} + \int_6^7 \frac{dx}{(x-6)^2} + \int_7^{+\infty} \frac{dx}{(x-6)^2} \\
&= \lim_{r\to 6^-} \int_0^r \frac{dx}{(x-6)^2} + \lim_{s\to 6^+} \int_s^7 \frac{dx}{(x-6)^2} + \lim_{t\to+\infty} \int_7^t \frac{dx}{(x-6)^2}.
\end{aligned}$$

In order for our original integral to converge, we need all three of these to converge. But we see that

$$\begin{aligned}
\lim_{r\to 6^-} \int_0^r \frac{dx}{(x-6)^2} &= \lim_{r\to 6^-} \left(\frac{-1}{x-6}\right)\Bigg|_0^r \\
&= \lim_{r\to 6^-} \frac{1}{6-r} - \frac{1}{6} = +\infty
\end{aligned}$$

diverges, so the whole integral diverges.

3.1.3 The Comparison Test for Improper Integrals

Sometimes we don't care much what the area of a region is; we only want to know if it's finite or not. In those cases this theorem is enough:

Theorem 3.16. *Suppose f and g are continuous functions with $f(x) \geq g(x) \geq 0$ for $x \geq a$. Then:*

- *If $\int_a^{+\infty} f(x)\,dx$ is convergent then $\int_a^{+\infty} g(x)\,dx$ is convergent.*
- *If $\int_a^{+\infty} g(x)\,dx$ is divergent then $\int_a^{+\infty} f(x)$ is divergent.*

This basically tells us that if the area under $f(x)$ is finite, then any area it contains must be finite; and if the area under $g(x)$ is infinite, any area containing it must be infinite.

Example 3.17. $\int_1^{+\infty} x^r\,dx$ is convergent if $r \leq -2$ since $x^r \leq x^{-2}$ on $[1, +\infty)$. $\int_1^{+\infty} x^r$ is divergent if $r \geq -1$ since $x^r \geq x^{-1}$ in that case.

Example 3.18. Does $\int_0^{+\infty} e^{-x^2}\,dx$ converge?

We will find this slightly easier if we split this up into two integrals. It's clear that $\int_0^1 e^{-x^2}\,dx$ converges because it is a finite proper integral of a continuous function. So we just have to show that $\int_1^{+\infty} e^{-x^2}\,dx$ converges. But for every x in $[1, +\infty)$, we know that $e^{-x^2} < 2xe^{-x^2}$, and we've shown that $\int_1^{+\infty} 2xe^{-x^2}\,dx$ converges. Thus by the comparison test, $\int_1^{+\infty} e^{-x^2}\,dx$ converges.

3.2 Geometric Applications

We can also use the integral to answer some other fun little geometry questions.

3.2.1 Arc Length

The first one we want to look at is the length of a curve.

$$L = \int_a^b \sqrt{1 + f'(x)^2}\,dx \tag{1}$$

Example 3.19. Let's take the curve $y^2 = x^3$ and find the arc length of the curve between $(0, 0)$ and $(4, 8)$.

We have that $y = \sqrt{x^3}$ on this curve, so $y' = \frac{3}{2}x^{1/2}$. Then

$$\begin{aligned}
L &= \int_0^4 \sqrt{1 + y'^2}\,dx \\
&= \int_0^4 \sqrt{1 + 9/4x}\,dx = \frac{2}{3} \cdot \frac{4}{9}(1 + 9/4x)^{3/2}\Big|_0^4 \\
&= \frac{8}{27}\left(10^{3/2} - 1\right).
\end{aligned}$$

Example 3.20. Let $f(x) = x^2$. Let's find the arc length between $x = 0$ and $x = 4$.

We have

$$L = \int_0^4 \sqrt{1 + (2x)^2}\, dx$$

We can set $2x = \tan\theta$, so $dx = \frac{1}{2}\sec^2\theta\, d\theta$. This gives us

$$\begin{aligned} L &= \int_0^4 \frac{1}{2}\sqrt{1+(2x)^2}\, dx = \int_0^{\arctan 8} \frac{1}{2}\sqrt{1+\tan^2(\theta)}\sec^2\theta\, d\theta \\ &= \int_0^{\arctan 8} \frac{1}{2}\sec^3\theta\, d\theta \end{aligned}$$

and at this point I...give up on this integral. You can look it up, or you can plug it into a computer algebra package. We get

$$\frac{1}{4}\sec(x)\tan(x) + \frac{1}{2}\ln|\sec(x) + \tan(x)|\Big|_0^{\arctan 8} = \approx 16.819.$$

Somtimes it's as easy—or easier—to integrate with respect to y.

Example 3.21. Consider the graph of the hyperbola $xy = 1$ as y varies from 1 to 3. What is the arc length of this curve?

We could view this as a function of x: $y = 1/x$, so $y' = -1/x^2$, and then

$$L = \int_{1/3}^1 \sqrt{1 + 1/x^4}\, dx \approx 2.14662.$$

Alternatively, we could view it as a function of y: $x = 1/y$, so $x' = -1/y^2$, and we have

$$L = \int_1^3 \sqrt{1 + 1/y^4}\, dy \approx 2.14662.$$

Which one is more convenient depends on what you're doing.

3.2.2 Surface Area

We can also try to compute the surface area of a surface produced by revolving a curve an axis. This is a lot like computing the area of a solid of revolution, but now we only want to look at the area on the outside.

There are a couple of ways to think about this formula, but they both get you to essentially the same place. We can imagine cutitng the surface into little strips, and then pretending these strips are small cylinders. The radius of the cylinder is given by the height of the function; the height of the cylinder is given by the *arc length* of the bit of the function inside the band. (When the function is steeper, the average radius can be the same, but the width of our imaginary band is much greater.)

Thus the area of one band will be the circumference of the circle, which is $2\pi f(x)$, times the width of the band, which from section 3.2.1 we know is approximately $\sqrt{1+f'(x)^2}dx$. Thus the surface area of a surface of revolution is

$$SA = \int_a^b 2\pi f(x)\sqrt{1+f'(x)^2}\,dx. \tag{2}$$

Example 3.22. What is the surface area of a sphere of radius 1? We can look at this as taking the curve $\sqrt{1-x^2}$ on $[-1,1]$ and revolving it around the x axis. Since $f'(x) = \frac{-x}{\sqrt{1-x^2}}$, we get

$$\begin{aligned} SA &= \int_{-1}^{1} 2\pi\sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}}\,dx \\ &= 2\pi \int_{-1}^{1} \sqrt{1-x^2}\sqrt{\frac{1-x^2+x^2}{1-x^2}}\,dx \\ &= 2\pi \int_{-1}^{1} \sqrt{1-x^2}\sqrt{\frac{1}{1-x^2}}\,dx \\ &= 2\pi \int_{-1}^{1} 1\,dx = 4\pi. \end{aligned}$$

But we could also compute the area of a part of the sphere, say the band in the middle. Then we'd have

$$\begin{aligned} SA &= \int_{-1/2}^{1/2} 2\pi\sqrt{1-x^2}\sqrt{1+\frac{x^2}{1-x^2}}\,dx \\ &= 2\pi \int_{-1/2}^{1/2} \sqrt{1-x^2}\sqrt{\frac{1-x^2+x^2}{1-x^2}}\,dx \\ &= 2\pi \int_{-1/2}^{1/2} \sqrt{1-x^2}\sqrt{\frac{1}{1-x^2}}\,dx \\ &= 2\pi \int_{-1/2}^{1/2} 1\,dx = 2\pi. \end{aligned}$$

So the middle half of the sphere has exactly half the surface area of the whole sphere!

Example 3.23. Let $f(x) = \sqrt[3]{3x}$. Take the portion of the graph where $0 \leq y \leq 2$ and rotate it around the y axis. What is the surface area?

This one will be easiser, for multiple reasons, to view as a function of y. So we have

$y = \sqrt[3]{3x}$ and thus $x = y^3/3$. Then $x' = y^2$, and we have

$$\begin{aligned} SA &= \int_0^2 \frac{2\pi y^3}{3}\sqrt{1+y^4}\,dy \\ &= \frac{2\pi}{3}\int_0^2 y^3\sqrt{1+y^4}\,dy \\ &= \frac{2\pi}{3}\frac{2}{12}(1+y^4)^{3/2}\Big|_0^2 \\ &= \frac{\pi}{9}\left(17^{3/2}-1\right) \approx 24.118. \end{aligned}$$

And we can finish up with my favorite application/paradox, combining area, surface area, and improper integrals.

Example 3.24 (Gabriel's Trumpet/Infinite Paint Can)**.** Consider a trumpet-shaped container, given by taking the curve $y = 1/x$ and rotating around the x-axis, for $x \geq 1$. We're going to imagine this as a giant, oddly-shaped paint can.

We can work out the volume of this shape fairly easily, using cross-sections. If we take cross-sections perpendicular to the x-axis, each cross section is a circle of radius $1/x$. The area of this circle will be $\frac{\pi}{x^2}$ and thus the total volume will be

$$\begin{aligned} \int_1^\infty \frac{\pi}{x^2}\,dx &= \lim_{t\to\infty}\int_1^t \frac{\pi}{x^2}\,dx \\ &= \lim_{t\to\infty}\frac{-\pi}{x}\Big|_1^t = \frac{-\pi}{t} - \frac{-\pi}{1} = \pi. \end{aligned}$$

Thus the volume of our paint can is π; the can can hold π gallons of paint.

But now let's imagine painting the paint can. How much paint would we need to cover it? What's the surface area of the can?

We can do our surface area setup. We have $f(x) = 1/x$ so that $f'(x) = -1/x^2$. Then the surface area is

$$\begin{aligned} \int_1^\infty \frac{2\pi}{x}\sqrt{1+(-1/x^2)^2}\,dx &= \lim_{t\to\infty}\int_1^t \frac{2\pi}{x}\sqrt{1+1/x^4}\,dx \\ &\geq \lim_{t\to\infty} 2\pi\int_1^t \frac{1}{x}\,dx \\ &= 2\pi\lim_{x\to\infty}\ln|x|\Big|_1^t \\ &= \lim_{t\to\infty} 2\pi(\ln|t|-1) = \infty. \end{aligned}$$

So the surface area of the paint can is infinite! You can fill the entire can with π gallons of paint, but it would take an infinite amount of paint to cover the interior of the paint can.

3.3 Differential Equations

Now that we have assigned derivatives a physical meaning, or even a large number of physical meanings, we can reason about how they interact with physical systems. In particular, we can make statements about the derivative of some function and how it relates to the values of that function.

Definition 3.25. A *differential equation* is an equation that relates the derivatives of a function to the values of that function.

For a simple example, consider the phrase "acceleration is proportional to force." Recall that acceleration is the second derivative of position. If force is itself a function of position, this translates to a differential equation, relating $f''(x)$ to $f(x)$.

Example 3.26 (Hooke's Law)**.** Hooke's law tells us that the force a fall exerts is proportional to the displacement of the fall; that is, for any given fall there is some constant k such that $F(t) = -kx(t)$, where $x(t)$ is the function that takes in the time and outputs the x coordinate of the fall. Since $F(t) = ma(t) = mx''(t)$, this gives us the differential equation $mx''(t) = -kx(t)$ or

$$x''(t) = -\frac{k}{m}x(t).$$

For simplicity let's assume $k = m$ so we have $x''(t) = -x(t)$.

Can we find a solution for this? We can start with the really silly or "trivial" solution. If the fall starts at neutral, it will never move, so we'd expect $x(t) = 0$. And indeed it is: $0'' = 0 = -0$, so the funcion $x(t) = 0$ is a solution to this differential equation.

Can we find a solution that involves any motion at all? We're looking for a function where $x''(t) = -x(t)$. And we actually know two of these: $x(t) = \sin(t)$ and $x(t) = \cos(t)$ both satisfy this differential equation. And this is why the equation for "simple harmonic motion" is built up out of sin and cos functions.

There are many different solutions we can use; for example, $3\sin(t) + 5\cos(t) = 17$ is a solution to this differential equation. It's easy to see that if a and b are any constants, then $x(t) = a\sin(t) + b\cos(t)$ is a solution to this differential equation. It's much less obvious, but true, that any solution to the Hooke's Law equation must have this form; even the trivial solution is given by $x(t) = 0\sin(t) + 0\cos(t)$.

To pick out the specific solution we need to know "initial conditions" that tell us the starting position. But if we know the starting position and starting velocity of the fall, we can determine a and b and thus get an exact formula for $x(t)$.

Something very interesting has happened here! In grade school, we learned to do simple arithmetic, like being asked to compute $3 + 5$ and calculating 8. As we got to algebra, we were asked instead to *solve equations*. We would get formulas like $3+x = 8$ and try to figure out what x is. This is the same sort of question but backwards—instead of computing with known numbers, we have to figure out which numbers will make the calculation work.

In pre-calculus and so far in this course, we have done calculuations with functions. Plug a number into this function; graph this function; take the derivative of this function. But here we are being asked to solve equations whose answers are *functions*. The question is, which function satisfies the given relationship? And if we have a candidate answer, we can test it by plugging it into the differential equation and seeing if the equation we get is true.

Example 3.27 (Proportional Growth)**.** The simplest possible (non-trivial) differential equation is probably $p'(t) = kp(t)$. This tells us that the rate of change of something we're measuring is proportional to the current level of that thing.

This often comes up in the context of population growth. If we look at, say, a breeding population of rabbits, then the number of new rabbits born each year depends on the number of rabbits that are already alive: if we start with two rabbits, we won't end the year with two million. If each rabbit on average produces three new rabbits in a year, we might approximate the derivative by saying $\frac{dp}{dt} = 3p(t)$. That is, the change in the total population of rabbits is equal to three times the current number of rabbits.

In this case, if we start a year with 100 rabbits, then we have $p'(0) = 3p(0) = 300$ so we expect to get three hundred new rabbits, and end the year with 400. The next year we will get $p'(1) = 3p(1) = 1200$, so we get 1200 new rabbits and end the year with 1600 rabbits. The derivative is different each year, but the proportional growth rate is not.

Can we find a function that satisfies $p'(t) = 3p(t)$? And so far in this course, the answer is "not really". The trivial solution will still work, actually: if we start with zero rabbits, then we will always have zero rabbits, and it is true that $0' = 3 \cdot 0$. But if we want a non-trivial solution, none of the functions we've seen so far will work here. In section 3.5.1 we will see that the solutions to this differential equation look like $p(t) = Ca^t$ for some constants C and a; a depends on the breeding rate, and C is the initial population of rabbits.

But this equation describes more than just rabbit population growth. Other cases where this equation appears include:

- Economic growth: the economy grows by 3% a year, so we have $p'(t) = .03p(t)$.
- Interest: if you are paying 8% interest per year, then your debt increases at a rate $d'(t) = .08d(t)$.

- Radioactive decay: some fraction of your sample of uranium will decay every year, so you have $u'(t) = ku(t)$. In this case k will be negative since your amount of uranium is decreasing.
- Heat transfer: the rate at which heat flows from a hot object to a cold object is proportional to the difference in temperature, so we have $T'(t) = kT(t)$.

Example 3.28 (Evans price change model)**.** Economists often use systems of differential equations to describe how the economy changes over time.

If there is a shortage of some good, which means that more people want to buy than sell, the price will tend to increase so that fewer people want to buy, more people want to sell, and the market clears. But the price doesn't change immediately. The Evans model says that the price change is proportional to the size of the shortage: $\frac{dp}{dt} = k(D - S)$, where D is the quantity demanded and S is the quantity supplied. So if the shortage is bigger, the price will increase faster.

So far, this looks sort of like exponential growth. But it's importantly different, because the size of the shortage is not the same thing as the price! We need to ask how demand depends on price. A simple model says that $D(p) = a - bp$ and $S(p) = r + sp$, where a is the amount demanded when the price is zero and r is the amount supplied when the price is zero. Then $-b = \frac{dD}{dp}$ and $s = \frac{dS}{dp}$ are the elasticities of demand and supply.

Plugging this back into the original model gives

$$p'(t) = k(a - bp(t) - r - sp(t)) = k\left((a - r) - (b + s)p(t)\right).$$

From this we can see that the trivial solution where the price is zero doesn't actually work here. And this should make sense, because if the price is zero you expect more people to want to buy than to sell. We also notice that it doesn't matter what the demand or supply elasticities are individually; it only matters what their sum is. We can use this equation to estimate the way the price will change over time.

There is a rich and powerful theory for solving differential equations. We won't really be studying it in this course, since we don't have the tools to understand it; we would need integrals from calculus 2 and also a number of tools from linear algebra. But there are a few questions we can address.

First, we can check whether a given function actaully satisfies a given differential equation.

Example 3.29. Confirm that $f(x) = x^2 + x + 1$ satisfies $2f(x) - xf'(x) = x + 2$.

We compute $f'(x) = 2x + 1$, so $2f(x) - xf'(x) = 2x^2 + 2x + 2 - (2x^2 + x) = x + 2$.

Second, we can solve what are called "initial value problems" or "boundary value problems". A given differential equation will usually have infinitely many solutions, as with the solutions $a\sin(t) + b\cos(t)$ to the equation $x''(t) = -x(t)$. This tells us the general shape of the solution, but doesn't give us an actual solution.

As we discussed, the specific solution depends on where things start. On the Hooke's Law fall system, if your fall starts at neutral then it will never move; if it starts extremely displaced then it will oscillate wildly. So to know the position over time we need to know where the system starts, known as the *initial conditions*.

Example 3.30. Suppose we have a Hooke's Law system with $m = k$, so that we get the differential equation $x''(t) = -x(t)$. We said earlier that then $x(t) = a\sin(t) + b\cos(t)$ for some constants a and b.

Suppose now we start with the weight stationary and displaced by 1 meter. Since this is the starting conditions, this is at time 0, so this means that $x(0) = 1$ and $x'(0) = 0$. Now we have enough information to figure out a and b and find a specific solution to describe the path of our fall.

Since $x(0) = 1$ we know that

$$1 = a\sin(0) + b\cos(0) = b,$$

and since $x'(0) = 0$ we know that

$$0 = a\cos(0) - b\sin(0) = a$$

so we have $a = 0, b = 1$, and $x(t) = \cos(t)$.

And as we think about it, this answer makes some sense: there's no reason for the fall to ever displace further than one meter, and so that's exactly what we see here.

Sometimes instead of an initial value problem we have a boundary value problem. In a boundary value problem you get position values at different times, rather than position and velocity at the same time.

Example 3.31. Suppose we have a Hooke's Law setup with $m-k$, so $x(t) = a\sin(t)+b\cos(t)$.

Suppose we know that $x(0) = 2$ and $x(\pi/4) = \sqrt{8}$. Then we know that

$$\begin{aligned} a\sin(0) + b\cos(0) &= 2 \\ b &= 2 \\ a\sin(\pi/4) + b\cos(\pi/4) &= \sqrt{8} \\ a\sqrt{2}/2 + 2\cdot\sqrt{2}/2 &= \sqrt{8} \\ a/2 + 1 &= 2 \\ a &= 2. \end{aligned}$$

Thus we have that $x(t) = 2\sin(t) + 2\cos(t)$.

Notice that in either of these cases, we need to take only two measurements to know exactly what happens at every possible time. This is because our differential equation, coming from a physical law, severely constrains what our answers can possibly look like; we only need a bit more information to have it nailed down precisely, one measurement for each constant.

Of course, in the real world, measurements come with errors so we need to take more than two. But we can get a lot of information from our differential equation telling us what sort of relationships to look for.

Example 3.32. Suppose $f(x) = ax^2 + bx + c$ is a polynomial satisfying some differential equation, and we have $f(0) = 0, f'(0) = 1, f''(0) = 2$. What can we say about $f(x)$?

We see that $f(0) = c = 0$, $f'(x) = 2ax + b$ so $f'(0) = b = 1$, and $f''(x) = 2a$ so $f''(0) = 2a = 2$. Thus $a = b = 1$ and $c = 0$, so $f(x) = x^2 + x$.

Example 3.33. Suppose $g(x) = ax^2 + bx + c$ is a polynomial satisfying some differential equation, with $g(1) = 2, g'(2) = 3, g''(3) = 4$. What can we say about g?

We have $g(1) = a + b + c$. $g'(x) = 2ax + b$ so $g'(2) = 4a + b = 3$, and $g''(x) = 2a$ so $g''(3) = 2a = 4$. Thus we have $a = 2$. Going back to g' we see that $8 + b = 3$ so $b = -5$. Then plugging into g we have $2 - 5 + c = 2$ so $c = 5$. Thus $g(x) = x^2 - 5x + 5$.

3.4 Separable differential equations

Even ordinary differential equations are hard to solve; in general solving them involves using power series and Fourier series (which are series in $\sin(2\pi nx)$ rather than in x^n). But we can solve a special type of differential equation.

Definition 3.34. A *separable differential equation* is a differential equation that can be written
$$\frac{dy}{dx} = g(x)f(y)$$
for some functions g and f.

We call these separable because we can separate the variables, by putting all the ys on one side and all the xs on the other. Heuristically, we divide by $f(y)$ and "multiply by dx": this gives us
$$\frac{dy}{f(y)} = g(x)\,dx$$
and we can now integrate both sides. We can justify this via the chain rule: if
$$\int \frac{dy}{f(y)} = \int g(x)\,dx$$
then
$$\begin{aligned} \frac{d}{dx}\left(\int \frac{dy}{f(y)}\right) &= \frac{d}{dx}\left(\int g(x)\,dx\right) \\ \frac{dy}{dx}\cdot\frac{d}{dy}\left(\int \frac{dy}{f(y)}\right) &= \frac{d}{dx}\left(\int g(x)\,dx\right) \\ \frac{dy}{dx}\cdot\frac{1}{f(y)} &= g(x). \end{aligned}$$

Example 3.35. Solve $y' = x/y$ for the initial value $y(0) = 2$.

We have
$$\begin{aligned} \frac{dy}{dx} &= \frac{x}{y} \\ \int y\,dy &= \int x\,dx \\ \frac{y^2}{2} &= \frac{x^2}{2} + C \\ y &= \pm\sqrt{x^2 + 2C} \end{aligned}$$

We can at this point replace the $2C$ with a C without losing anything, though the textbook suggets changing the variable to a K to make sure you don't confuse yourself. Either way, our initial condition is $y(0) = 2$, so we have $\pm\sqrt{0 + 2C} = 2$ and thus $C = 2$ and our square root must be positive. So we have $y = \sqrt{x^2 + 4}$.

Note that if our initial condition were negative, say $y(0) = -2$, then we'd have a negative square root instead, and $y = -\sqrt{x^2 + 4}$.

Example 3.36. Solve $y' = \frac{3x+\cos x}{2y+y^2}$ for the initial condition $y(0) = 3$.

We have

$$\begin{aligned} \frac{dy}{dx} &= \frac{3x + \cos x}{2y + y^2} \\ \int 2y + y^2\,dy &= \int 3x + \cos x\,dx \\ y^2 + \frac{y^3}{3} &= \frac{3x^2}{2} + \sin x + C. \end{aligned}$$

This gives y as a function of x implicitly, but there's no good way to write it explicitly as a function of x. We can, however, work out the constant; we have $3^2 + 3^3/3 = 0 + 0 + C$ and thus $C = 18$. So our solution is

$$y^2 + \frac{y^3}{3} = \frac{3x^2}{2} + \sin x + 18.$$

Example 3.37. Find the general solution to $(y^2 + xy^2)y' = 1$.

We have

$$\begin{aligned} \int y^2 dy &= \int \frac{dx}{1+x} \\ \frac{y^3}{3} &= \ln|1+x| + C \\ y^3 &= \sqrt[3]{3\ln|1+x| + 3C}. \end{aligned}$$

Example 3.38. Find the specific solution to $P' = \sqrt{Pt}$ with $P(0) = 2$.

We have

$$\begin{aligned} \int \frac{dP}{\sqrt{P}} &= \int \sqrt{t}\,dt \\ 2\sqrt{P} &= \frac{2}{3}t^{3/2} + C \\ P &= \left(t^{3/2}/3 + C/2\right)^2 \end{aligned}$$

and since $P(1) = 2$ we have $2 = (0 + C/2)^2$ and thus $C = 2\sqrt{2}$. So the specific solution is

$$P = \left(\frac{t^{3/2}}{3} + \sqrt{2}\right)^2.$$

3.5 Some common separable differential equations

3.5.1 Exponential Growth/Decay

This is often covered in Calculus 1. A common scenario is when the rate of change of a variable is proportional to its current level; for instance, if we have a radioactive substance, a constant *fraction* of it will decay in any given time interval. (i.e. there is a time T such that every T seconds, half of the substance decays. This number is the "half-life").

We can capture this idea with the differential equation $y' = ky$ for some constant k: the rate of change of y is proportional to the current level of y.

To solve this we compute $\frac{dy}{dx} = ky \Rightarrow \frac{dy}{y} = k\,dx \Rightarrow \ln|y| = kx + C$. Raising e to the power of both sides gives $y = e^{kx+C}$, and after changing the constant we have $y = Ce^{kx}$. In particular, we see $y = C$ when $x = 0$, so y would be the amount of substance that exists at time $x = 0$.

Example 3.39. The half-life of Radium-226 is 1590 years. How much of a $100g$ mass of radium will be left after 1000 years?

This is a differential equation $y' = ky$ with initial value $y(0) = 100$, and we also need to find the constant k. In order to do that, we use the "boundary condition" $y(1590) = 50$. This gives us that

$$\begin{aligned} 100 &= Ce^{k\cdot 0} & \Rightarrow C &= 100 \\ 50 &= Ce^{k\cdot 1590} & 50 &= 100e^{k\cdot 1590} \\ \frac{1}{2} &= e^{k\cdot 1590} & \ln(1/2) &= 1590k \\ \ln(1) - \ln(2) &= 1590k & \frac{-\ln(2)}{1590} &= k. \end{aligned}$$

This tells us that

$$y(t) = 100e^{\frac{-t\ln(2)}{1590}}.$$

Thus

$$y(1000) = 100e^{\frac{-1000\ln(2)}{1590}} \approx 100e^{-.436} \approx 64.7g.$$

So after 100 years we will have 64.7 g of radium-226 left. This should pass a sanity check, since it will take us 1590 years to get to 50.

Example 3.40. Let $P(t)$ be the size of a population of animals or people or Tribbles at time t. In the absence of resource restrictions, the population will grow at a rate $\frac{dP}{dt} = kP(t)$, where k is some constant representing the rate of growth. Then we compute

$$\int \frac{dP}{P} = \int k\,ct$$

and thus we have $\ln(P) = kt + C$ and $P(t) = Ce^{kt}$. What is C? If we have initial condition $P(0) = P_0$ then we see that $C = P_0$, the level of the population at time $t = 0$.

The total population of the world was 3 billion people in 1960, and 4 billion in about 1975. Setting $t = 0$ to be 1960 and fitting this to our model, we have: $Ce^{k0} = 3$ and $Ce^{15k} = 4$. Thus we must have $C = 3$, and then $e^{15k} = 4/3$ implies that $15k = \ln(4/3)$ and so $k = \ln(4/3)/15$. If we want to estimate global population in 2020, this gives us

$$P(60) = 3 \cdot e^{60 \cdot \ln(4/3)/15} = 3 \cdot e^{4 \ln(4/3)} = 3 \cdot (4/3)^4 \approx 9.48.$$

(Actual estimates put it at 7.7 billion, because population growth has been leveling off).

Now let's use our model to estimate when the population will reach 12 billion. We want

$$12 = 3 \cdot e^{t \ln(4/3)/15} \tag{3}$$

$$4 = \left(\frac{4}{3}\right)^{t/15} \tag{4}$$

$$\log_{4/3} 4 = t/15 \tag{5}$$

$$15 \frac{\ln(4)}{\ln(4/3)} = t \tag{6}$$

$$72.3 \approx t \tag{7}$$

So our model predicts that the world's population will reach 12 billion in about 2032.

3.5.2 Mixing Problems

A slightly more complicated variant on exponential growth occurs when there is constant growth and exponential decay at the same time (or vice versa).

Example 3.41. Suppose we have a tank containing 10 kg of salt dissolved in 1000 L of water. We can pump in a brine solution of .02 kg of salt per liter of water at 10L per minute, while ten L of solution drains out of the tank each minute. How much salt is in the tank after twenty minutes?

Let y be the amount of salt in the tank. Then we have $y(0) = 10$, and we have $y' = .2 - \frac{y}{100}$, since each minute the tank is gaining .1 kg of salt and losing a hundredth of its salt content.

We can easily rewrite this as

$$\frac{dy}{dt} = \frac{20 - y}{100}$$
$$\int \frac{dy}{20 - y} = \int \frac{dt}{100}$$
$$\ln|20 - y| \cdot (-1) = \frac{t}{100} + C$$

Since $y(0) = 10$ so we have $-\ln(10) = C$, so we get

$$\begin{aligned} -\ln|20-y| &= \frac{t}{100} - \ln(10) \\ |20-y| &= 10e^{-t/100} \\ 20-y &= 10e^{-t/100} \\ y &= 20 - 10e^{-t/100}. \end{aligned}$$

Thus after 20 minutes we have

$$y = 20 - 10e^{-1/5} \approx 11.81kg.$$

3.5.3 Logistic Growth

Earlier we talked about exponential growth, which describes, among other things, population growth when there are no resource constraints. But most populations cannot actually grow exponentially without bound: eventually they run out of space or food or some other resource.

A simple model for this is the model of *logistic growth*. Let M be the *carrying capacity*, i.e. the maximum population. Then when our population is small, we want growth roughly proportional to the size of our population, as before. But as the population gets closer to M the rate of growth gets closer to 0; a simple equation that captures this is the *logistic differential equation* first developed by Pierre-François Verhulst in the 1840s:

$$\frac{dy}{dt} = ky(M-y).$$

This is separable, so we can write

$$\begin{aligned} \int k\,dt &= \int \frac{dy}{y(M-y)} \\ &= \int \frac{1}{M}\left(\frac{1}{y} - \frac{1}{M-y}\right) dy \\ &= \frac{1}{M}\left(\int \frac{dy}{y} - \frac{dy}{M-y}\right) \\ kt + C &= \frac{1}{M}\left(\ln|y| - \ln|M-y|\right). \end{aligned}$$

Since $0 < y < M$ both y and $M-y$ are positive, so this gives

$$M(kt+C) = \ln\left(\frac{y}{M-y}\right)$$

and thus

$$\frac{y}{M-y} = Ae^{Mkt}.$$

Given an initial condition $y(0) = y_0$ we have

$$\frac{y}{M-y} = \frac{y_0}{M-y_0}e^{Mkt}$$

and solving for y to write y as a function of t gives us

$$\begin{aligned}
y &= (M-y)\frac{y_0}{M-y_0}e^{Mkt} \\
&= \frac{My_0}{M-y_0}e^{Mkt} - \frac{yy_0}{M-y_0}e^{Mkt} \\
y\left(1 + \frac{y_0}{M-y_0}e^{Mkt}\right) &= \frac{My_0}{M-y_0}e^{Mkt} \\
y &= \frac{\frac{My_0}{M-y_0}e^{Mkt}}{1 + \frac{y_0}{M-y_0}e^{Mkt}} \\
&= \frac{My_0e^{Mkt}}{M - y_0 + y_0e^{Mkt}} \\
&= \frac{My_0}{(M-y_0)e^{-Mkt} + y_0}.
\end{aligned}$$

We can see that, as expected, as $t \to +\infty$ we have $y \to M$.

Example 3.42. Let's look back at our population growth table. Suppose

$$\frac{dP}{dt} = kP(M-P)$$

and thus

$$P = \frac{MP_0}{(M-P_0)e^{-Mkt} + P_0}.$$

We know from earlier comments that $P_0 = P(0) = 3$ if 0 corresponds to 1960. We also have $P(15) = 4$ and $P(60) = 7.7$. We need to solve for M and for k, and it's easier to start with k. We have

$$\begin{aligned}
4 &= \frac{M \cdot 3}{(M-3)e^{-Mk\cdot 15} + 3} \\
3M(4M &- 12)e^{-15Mk} + 12 \\
e^{-15Mk} &= \frac{3M-12}{4M-12} \\
-15Mk &= \ln\left|\frac{3M-12}{4M-12}\right| \\
k &= \frac{-1}{15M}\ln\left|\frac{3M-12}{4M-12}\right|.
\end{aligned}$$

We now need to solve for M. We'll use the fact that the world population in 2014 was 7.2 billion. Solving by hand would be terribly annoying, but I asked Mathematica and it told me that if $P(54) = 7.7$ then $M \approx 13.2$. Plugging this in gives us

$$P(t) = \frac{39.6}{10.2e^{-.026t} + 3}$$

which tells us $P(0) = 3, P(15) = 4, P(54) = 7.2$, and $P(60) = 7.7$ in line with current projections. Using this equation we can ask again when population will reach 12 billion; we see this happens when $t = 135$, or in about 2095. (Current projections are about 11 billion for 2100; this model is better than our previous one, but still not quite as good as you can do with a full-time team of statisticians).