# 4 Sequences and Series

## 4.1 Sequences

### 4.1.1 Sequences and Limits

**Definition 4.1.** A *sequence* of real numbers is a (usually infinite) ordered list of real numbers. We write $(a_n)_{n=1}^{\infty}$ for the sequence

$$(a_1, a_2, a_3, \dots)$$

where each $a_n$ is a real number.

*Remark* 4.2. We can think of a sequence as a function from the natural numbers to the real numbers. $f(n)$ is the $n$th element of the sequence. This isn't always useful, though.

**Example 4.3.** A few examples of sequences. Some of these will look familiar:

(a) $(1, 1, 1, 1, 1, \dots)$

(b) $(1, 2, 3, 4, \dots)$

(c) $(2^{10}, 17, \sqrt[5814]{3^{11} - 1}, 1, 1, 1, \dots)$

(d) $(1, 1, 2, 3, 5, 8, 13, \dots)$

(e) $(1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots)$

(f) $(3, 3.1, 3.14, 3.141, 3.1415, \dots)$

(g) $(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots)$

(h) $(1, \frac{\sqrt{3}}{2}, \frac{1}{2}, 0, -\frac{1}{2}, -\frac{\sqrt{3}}{2}, -1, \dots)$

In most of these sequences the pattern is pretty obvious. In sequence (a) we have $a_n = 1$. In sequence (b) we have $a_n = n$ and in sequence (e) we have $a_n = 1/n$. Less obviously, in sequence (g) we have $\frac{n}{n+1}$ and in (h) we have $\cos(n\pi/6)$.

In fact, not all sequences have nice descriptions like this. Sequence (d) is the *fibonacci sequence*, which is defined "inductively" or "recursively" by $f_1 = 1, f_2 = 1, f_n = f_{n-1} + f_{n-2}$ for $n \geq 3$. (This sequence was originally defined to work on problems about rabbit-breeding; it appears often in nature). Even worse are sequences like (c) which show no particular pattern at all; these are still sequences.

**Example 4.4.** What is the general form of the sequence $(1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \dots)$? We see that $a_n = \frac{1}{n^2}$.

What about $(1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \dots)$ ? $a_n = \frac{1}{2^n}$.

We can see that some of these sequences look like they're "going somewhere"—in fact, sequence (a) is there already. But sequences (e) and (f) seem to be getting closer and closer to some value.

When the terms of a sequence are getting closer and closer to some value, we say that it has a *limit*. In particular, we say the sequence $(a_n)$ has a limit $L$ in the real numbers if we can make the numbers $a_n$ get as close to $L$ as we want just by taking $n$ to be sufficiently big.

**Example 4.5.** The sequence $(1/n^2)$ has a limit of 0.

The sequence $1/2^n$ also have a limit of 0.

Now, if you're like me, this definition is uncomfortably fuzzy. Fortunately (or unfortunately, depending on your taste), we can make it much more precise:

**Definition 4.6.** Let $(a_n)$ be a sequence of real numbers. We say that $(a_n)$ has a limit $L$, and write $\lim_{n\to+\infty} a_n = L$, if, for every real number $\epsilon > 0$, there is a natural number $N$ such that, whenever $n \geq N$, $|a_n - L| < \epsilon$.

If a sequence has a limit in the real numbers we say the sequence *converges*. Otherwise we say the sequence *diverges*, and the limit *does not exist*.

**Example 4.7.** Prove that $\lim_{n\to+\infty} \frac{1}{n} = 0$.

Fix some $\epsilon > 0$. Then let $N > 1/\epsilon$. If $n \geq N$ then $\frac{1}{n} \leq \frac{1}{N} < \epsilon$, and thus $|a_n - 0| < \epsilon$. So by definition, $\lim_{n\to+\infty} \frac{1}{n} = 0$.

**Example 4.8.** Prove that $\lim_{n\to+\infty} \frac{n}{n+1} = 1$.

Let $\epsilon > 0$. Then let $N > 1/\epsilon$. If $n \geq N$, then

$$\left|\frac{n}{n+1} - 1\right| = \left|\frac{-1}{n+1}\right| \leq \frac{1}{n+1} \leq \frac{1}{N} < \epsilon.$$

Thus by definition, $\lim_{n\to+\infty} \frac{n}{n+1} = 1$.

**Example 4.9.** Prove that $\lim_{n\to+\infty} \frac{1}{n^2} = 0$.

Let $\epsilon > 0$. Then let $N > 1/\sqrt{\epsilon}$. If $n \geq N$ then $|\frac{1}{n^2} - 0| = \frac{1}{n^2} \leq \frac{1}{N^2} < \epsilon$. So by definition, $\lim_{n\to+\infty} \frac{1}{n^2} = 0$.

**Example 4.10.** Prove that $\lim_{n\to+\infty}(-1)^n$ does not exist.

Heuristically, we notice that this sequence "bounces around"; it doesn't get closer to just one value. We can make this rigorous:

For a limit to exist, a certain statement needs to be true for any positive real number. So to prove that a limit does not exist we just need to find *one* real number for which the statement is false.

So let $\epsilon = 1$, and suppose a limit $L$ exists. Then we can find a $N$ such that if $n \geq N$, then $|(-1)^n - L| < 1$. In particular, we can find both even $n$ and odd $n$, and so it must be the case that $|1 - L| < 1$ and $|-1 - L| < 1$. But there is no number $L$ that makes this true. So no limit exists.

### 4.1.2 Limit Laws

Computing limits in this way is important, and a good exercise, but sometimes a bit painful. We'd like a way to compute limits that doesn't require us to do this every time.

**Proposition 4.11.** *If $(a_n)$ and $(b_n)$ are convergent sequences and $c$ is a constant, then*

- $\lim_{n\to+\infty} a_n \pm b_n = \lim_{n\to+\infty} a_n \pm \lim_{n\to+\infty} b_n$

  *Proof.* Let $\epsilon > 0$. Then there are $N_a, N_b > 0$ such that if $n \geq N_a$ then $|a_n - L_a| < \epsilon/2$ and if $n \geq N_b$ then $|b_n - L_b| < \epsilon/2$. Let $N \geq N_a, N_b$. Then if $n \geq N$, we have

  $$\begin{aligned} |(a_n + b_n) - (L_a + L_b)| &= |(a_n - L_a) + (b_n - L_b)| \\ &\leq |a_n - L_a| + |b_n - L_b| < \epsilon/2 + \epsilon_2 = \epsilon. \end{aligned}$$

  □

- $\lim_{n\to+\infty} c = c$

  *Proof.* Let $\epsilon < 0$ and let $N = 1$. Then if $n \geq n$, we have $|c - c| = 0 < \epsilon$. □

- $\lim_{n\to+\infty} ca_n = c \lim_{n\to+\infty} a_n$

  *Proof.* If $c = 0$ this is obvious (by the previous fact). So assume $c \neq 0$.

  Let $\epsilon > 0$. Then there is a $N > 0$ such that if $n \geq N$, then $|a_n - L| < \epsilon/|c|$. So if $n \geq N$, then

  $$|ca_n - cL| = |c| \cdot |a_n - L| < |c| \cdot (\epsilon/|c|) = \epsilon.$$

  □

- $\lim_{n\to+\infty} a_n b_n = \lim_{n\to+\infty} a_n \lim_{n\to+\infty} b_n$
- $\lim_{n\to+\infty} \frac{a_n}{b_n} = \frac{\lim_{n\to+\infty} a_n}{\lim_{n\to+\infty} b_n}$ *if* $\lim_{n\to+\infty} b_n \neq 0$.
- $\lim_{n\to+\infty} a_n^p = (\lim_{n\to+\infty} a_n)^p$ *if* $p, a_n > 0$.

**Example 4.12.** What is $\lim_{n\to+\infty} \frac{n+1}{n}$?

We can write

$$\begin{aligned} \lim_{n\to+\infty} \frac{n+1}{n} &= \lim_{n\to+\infty} (1 + \frac{1}{n}) \\ &= \lim_{n\to+\infty} 1 + \lim_{n\to+\infty} \frac{1}{n} = 1 + 0 = 1. \end{aligned}$$

**Example 4.13.** What is the limit of the sequence $\sqrt{n+1} - \sqrt{n}$?

We conjecture that $\lim_{n\to\infty} \sqrt{n+1} - \sqrt{n} = 0$, so we want to bound the quantity $|\sqrt{n+1} - \sqrt{n} - 0|$. Using a familiar trick from Calculus 1, we see

$$\begin{aligned}\sqrt{n+1} - \sqrt{n} &= \frac{(\sqrt{n+1} + \sqrt{n})(\sqrt{n+1} - \sqrt{n})}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} \\ &= \frac{1}{\sqrt{n+1} + \sqrt{n}}.\end{aligned}$$

Thus

$$\lim_{n\to+\infty} \sqrt{n+1} - \sqrt{n} = \lim_{n\to+\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} = 0.$$

That last step was admittedly a bit fuzzy. The basic idea is that our sequence "looks like" $\frac{1}{\sqrt{n}}$; in particular, our sequence is smaller than $\frac{1}{\sqrt{n}}$ and $\frac{1}{\sqrt{n}} \to 0$, so our sequence should also get close to zero. We can make that precise with the Squeeze Theorem:

**Theorem 4.14** (Squeeze Theorem)**.** *If $a_n \leq b_n \leq c_n$ for $n \geq n_0$ and $\lim_{n\to+\infty} a_n = \lim_{n\to+\infty} c_n = L$ then $\lim_{n\to+\infty} b_n = L$.*

To continue the earlier example, we have

$$\begin{aligned}\lim_{n\to+\infty} \sqrt{n+1} - \sqrt{n} &= \lim_{n\to+\infty} \frac{1}{\sqrt{n+1} + \sqrt{n}} \\ 0 \leq \frac{1}{\sqrt{n+1} + \sqrt{n}} &\leq \frac{1}{\sqrt{n}}.\end{aligned}$$

We know that $\lim_{n\to+\infty} 0 = 0$. And we can work out that

$$\begin{aligned}\lim_{n\to+\infty} \frac{1}{\sqrt{n}} &= \lim_{n\to+\infty} \left(\frac{1}{n}\right)^{1/2} \\ &= \left(\lim_{n\to+\infty} \frac{1}{n}\right)^{1/2} \\ &= 0^{1/2} = 0.\end{aligned}$$

Then by the squeeze theorem, $\lim_{n\to+\infty} \frac{1}{\sqrt{n+1}+\sqrt{n}} = 0$.

**Definition 4.15.** If $n$ is a natural number, we define $n$ *factorial*, written $n!$, to be

$$n! = n \cdot (n-1) \dots 2 \cdot 1.$$

This is the product of all positive integers less than or equal to $n$.

**Example 4.16.** What is $\lim_{n\to+\infty} \frac{n!}{n^n}$?

We calculate that

$$a_n = \frac{n!}{n^n} = \frac{n(n-1)(n-2)\dots(2)(1)}{n \cdot n \cdot n \dots n \cdot n} = \frac{1}{n} \cdot \frac{n(n-1)(n-2)\dots(2)}{n^{n-1}}.$$

It's clear that the large fraction is between 0 and 1 since the numerator is positive, but smaller than the denominator. Thus we have $0 \leq a_n \leq \frac{1}{n}$, and $\lim_{n\to+\infty} 0 = \lim_{n\to+\infty} \frac{1}{n} = 0$. By the squeeze theorem, $\lim_{n\to+\infty} a_n = 0$.

**Example 4.17.** What is $\lim_{n\to+\infty} \frac{\sin n}{n}$?

This is a classic use case for the Squeeze Theorem. We know that $-1 \leq \sin n \leq 1$ for any $n$. So $\frac{-1}{n} \leq \frac{\sin n}{n} \leq \frac{1}{n}$ for any $n$. We know that $\lim_{n\to+\infty} \frac{1}{n} = 0$, and similarly $\lim_{n\to+\infty} \frac{-1}{n} = -\lim_{n\to+\infty} \frac{1}{n} = 0$. So by the squeeze theorem, $\lim_{n\to+\infty} \frac{\sin n}{n} = 0$.

These tools are all useful, but they don't work to do everything. There is one more trick we want to use. We spent a long time working on limits of functions in Calculus 1; we can use that work again.

**Theorem 4.18.** *Suppose $f(x)$ is a function such that $f(n) = a_n$ for every natural number $n$, and $\lim_{x\to+\infty} f(x) = L$. Then $\lim_{n\to+\infty} a_n = L$.*

*Remark* 4.19. This only works in one direction! If the function limit exists, then the sequence limit exists. But the converse is not true.

**Example 4.20.** What is $\lim_{n\to+\infty} \frac{n}{n+1}$? We see that if $f(x) = \frac{x}{x+1}$, then $f(n) = a_n$, so we can compute

$$\lim_{n\to+\infty} \frac{n}{n+1} = \lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{x/x}{(x+1)/x} = \lim_{x\to+\infty} \frac{1}{1+1/x} = 1.$$

Thus $\lim_{n\to+\infty} \frac{n}{n+1} = 1$.

**Example 4.21.** What is $\lim_{n\to+\infty} \frac{\ln n}{n}$?

We write $f(x) = \frac{\ln x}{x}$, and then $f(n) = a_n$. By L'Hôpital's rule, we have

$$\lim_{x\to+\infty} f(x) = \lim_{x\to+\infty} \frac{\ln x}{x} = \lim_{x\to+\infty} \frac{1/x}{1} = \lim_{x\to+\infty} \frac{1}{x} = 0.$$

Thus we also have that $\lim_{n\to+\infty} \frac{\ln n}{n} = 0$.

**Example 4.22.** What is $\lim_{n\to+\infty} \sin(n\pi)$?

Naively, we might argue this: Let $g(x) = \sin(x\pi)$. Then $\lim_{x\to+\infty} g(x)$ does not exist, since the function varies between $-1$ and $1$ no matter how large we let $x$ grow. Thus the limit does not exist.

However, our theorem only applies when $\lim_{x\to+\infty} g(x)$ exists; it tells us nothing if the limit of our function does not converge. In fact, for every $n$ we have $\sin(n\pi) = 0$, and thus

$$\lim_{n\to+\infty} \sin(n\pi) = \lim_{n\to+\infty} 0 = 0.$$

**Theorem 4.23.** *If* $\lim_{n\to+\infty} a_n = L$ *and* $f$ *is continuous at* $L$*, then*

$$\lim_{n\to+\infty} f(a_n) = f(L).$$

**Example 4.24.** What is $\lim_{n\to+\infty} \frac{(-1)^n}{n}$?

We have $\lim_{n\to+\infty} |a_n| = \lim_{n\to+\infty} \frac{1}{n} = 0$. Thus $\lim_{n\to+\infty} \frac{(-1)^n}{n} = 0$ as well.

We could also argue using the squeeze theorem: For all $n$ we have $\frac{-1}{n} \leq a_n \leq \frac{1}{n}$. Since $\lim_{n\to+\infty} \frac{-1}{n} = \lim_{n\to+\infty} \frac{1}{n} = 0$, we must have $\lim_{n\to+\infty} a_n = 0$.

**Corollary 4.25.** $\lim_{n\to+\infty} |a_n| = |\lim_{n\to+\infty} a_n|$ *if* $a_n$ *converges. If* $|a_n| \to 0$ *then* $a_n \to 0$.

*Proof.* Suppose $\lim_{n\to+\infty} |a_n| = L$, and let $\epsilon > 0$. Then there is a $N > 0$ such that for all $n \geq N$, we have

$$\begin{aligned} ||a_n| - 0| &< \epsilon \\ ||a_n|| &< \epsilon \\ |a_n| &< \epsilon \\ |a_n - 0| &< \epsilon. \end{aligned}$$

Thus when $n \geq N$ we have $|a_n - 0| < \epsilon$, and by definition $\lim_{n\to+\infty} a_n = 0$. □

### 4.1.3 Infinite limits

An important case is a sequence that gets "infinitely big." Importantly, these sequences diverge since they don't get close to a real number. However, we'd like to be able to talk about these specifically. We write $\lim_{n\to+\infty} a_n = +\infty$ if the terms $a_n$ get arbitrarily large as $n$ gets large. Similarly, $\lim_{n\to+\infty} a_n = -\infty$ if the terms $a_n$ get very negative as $n$ gets large. Sometimes it is convenient to write $\lim_{n\to\pm\infty} a_n = \pm\infty$ if the size of the terms gets large, but the sign changes.

**Definition 4.26.** We say that $\lim_{n\to+\infty} a_n = +\infty$ if for every $M > 0$ there is a $N > 0$ such that $a_n > M$ when $n > N$.

We say that $\lim_{n\to+\infty} a_n = -\infty$ if for every $M > 0$ there is a $N > 0$ such that $a_n < -M$ when $n > N$.

**Example 4.27.** Prove that $\lim_{n\to+\infty} n = +\infty$.

Let $M > 0$. Let $N > M$, and then if $n \geq N$, we have

$$a_n = n \geq N > M.$$

So by definition, $\lim_{n\to+\infty} n = +\infty$.

**Example 4.28.** Prove that $\lim_{n\to+\infty} -n^2 = -\infty$.

Let $M > 0$ and let $N < \sqrt{M}$. Then if $n \geq N$, we have

$$-n^2 < -N^2 < -(\sqrt{M})^2 = -M$$

so by definition $\lim_{n\to+\infty} -n^2 = -\infty$.

### 4.1.4 Completeness

We would like to say that every sequence either goes to infinity or has a (finite) limit. Unfortunately, this isn't the case, because a sequence can bounce up and down without ever settling on one value (remember $(-1)^n$). But if a sequence doesn't "bounce around" then we know it must either have a limit or go to infinity.

**Definition 4.29.** A sequence is *(monotonically) increasing* if $a_{n+1} \geq a_n$ for all $n$. A sequence is *(monotonically) decreasing* if $a_{n+1} \leq a_n$ for all $n$. In either case we say that such a sequence is *monotonic.*

A sequence is *bounded above* if there is an $A$ such that $a_n \leq A$ for all $n$. A sequence is *bounded below* if there is an $A$ such that $a_n \geq A$ for all $n$. A sequence that is bounded above and bounded below is *bounded.*

A monotone sequence doesn't bounce around; a bounded sequence doesn't go to infinity. In the real numbers, a sequence with both of these properties must have a limit.

**Fact 4.30.** *Every increasing sequence of real numbers that is bounded above converges to some real number. Every decreasing sequence of real numbers that is bounded below converges to some real number. In particular, every bounded monotonic sequence is convergent.*

*Remark* 4.31. The idea here is that every sequence that "should" have a finite limit does. If the terms get closer to each other, there is some limit they approach.

**Example 4.32.** $\sqrt{2}, \sqrt{2\sqrt{2}}, \sqrt{2\sqrt{2\sqrt{2}}}, \dots$.

If $0 \leq x \leq 2$ then $x \leq \sqrt{2x} \leq 2$. Thus since the first element is between 0 and 2, the sequence is increasing, and every element is $\leq 2$, so the sequence is bounded above by 2. Thus it must converge.

Can we see what it must converge to? If we look at the sequence $a_n^2/2$ we have $1, \sqrt{2}, \sqrt{2\sqrt{2}}, \dots$ and get the same sequence again, just "shifted by one." So

$$L = \lim_{n \to +\infty} a_n = \lim_{n \to +\infty} \frac{a_n^2}{2} = \frac{\left(\lim_{n \to +\infty} a_n\right)^2}{2} = \frac{L^2}{2}.$$

Thus $2L = L^2$ and $L = 2$.

Alternatively we can notice that $a_n = 2^{1-\frac{1}{2^n}}$. Then

$$\lim_{n \to +\infty} a_n = \lim_{n \to +\infty} 2^{1-\frac{1}{2^n}} = 2^{\left(\lim_{n \to +\infty} 1 - \frac{1}{2^n}\right)} = 2^{1-0} = 2.$$

**Example 4.33.** $r^n$ converges for what $r$?

First let's consider positive $r$. For positive $r$ we can use facts about the function $f_r(x) = r^x$; we know that $\lim_{x \to +\infty} f_r(x) = \lim_{x \to +\infty} r^x = 0$ if $0 < r < 1$, and it is $+\infty$ if $r > 1$. (It's easy to check that if $r = 1$ then $\lim_{n \to +\infty} 1^n = 1$, and similarly $\lim_{n \to +\infty} 0^n = 0$).

If $-1 < r < 0$ then we can use the corollary about absolute value: $0 < |r| < 1$ so $\lim_{n \to +\infty} |r^n| = \lim_{n \to +\infty} |r|^n = 0$, and thus $\lim_{n \to +\infty} r^n = 0$. If $r = -1$ then we have done this example already, and $\lim_{n \to +\infty} (-1)^n$ diverges. If $r < -1$ then $(r^n)$ diverges for similar reasons: all terms have absolute value greater than 1, and they alternate being positive and negative, and thus cannot possibly converge.

## 4.2 Series

In this section we will discuss a particular type of sequence called a series. Series are powerful and flexible tools that show up in many places in mathematics; they are used to compute approximations, they underlie integrals, and they are often used to solve differential equations.

But at base, we can think of a series as a sort of a discrete version of the integral. The integral is "continuous", which means it adds up values from every point in the domain; a series will add up the values at only distinct, separated points in the domain.

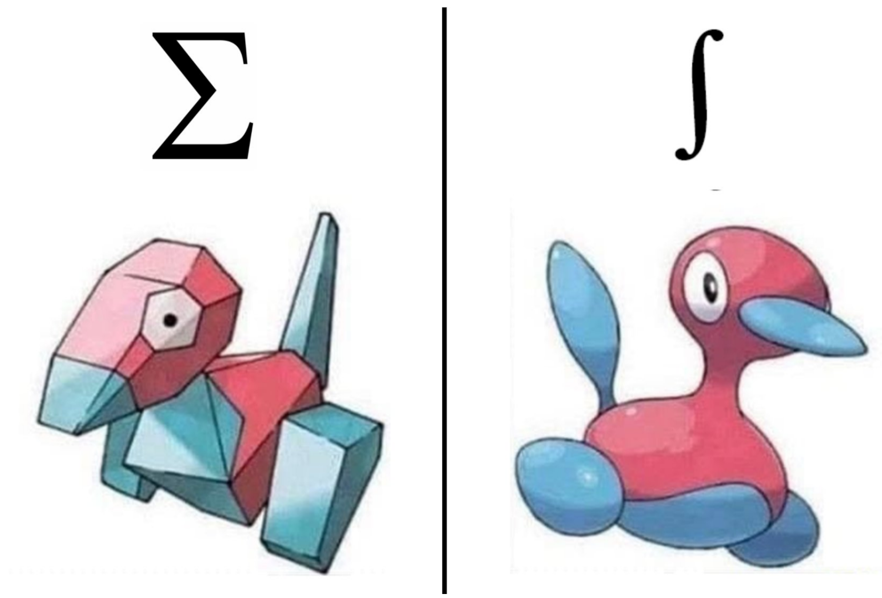


Figure 4.1: Meme courtesy of `@howie_hua` on Twitter

**Definition 4.34.** A *series* is a "sequence of partial sums." That is, a series is a sequence $(s_n)_{n=1}^{+\infty}$ where for some other real sequence $(a_n)$ we have

$$s_n = a_1 + a_2 + \cdots + a_n = \sum_{i=1}^{n} a_i.$$

If the sequence $(s_n)$ is convergent and $\lim_{n\to+\infty} s_n = s$, then we say the series $\sum a_n$ converges to $s$, which is the *sum* of the series. We write

$$\sum_{n=1}^{\infty} a_n = s \qquad \text{or} \quad a_1 + a_2 + \cdots + a_n + \cdots = s.$$

If $(s_n)$ is divergent, then the series is also divergent.

**Example 4.35.** A couple of the sequences we saw in the last section are "really" series.

- $1, 2, 3, \ldots$ can be viewed as $\sum_{i=1}^{\infty} 1$.
- Any infinite decimal representation is really a series: we have

$$\pi = 3 + 1 \cdot 10^{-1} + 4 \cdot 10^{-2} + 1 \cdot 10^{-3} + \ldots$$

**Example 4.36.** $\frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \ldots$ is the series $\sum_{i=1}^{\infty} \frac{1}{2^i}$. We see that the partial sum $s_n = \sum_{i=1}^{n} \frac{1}{2^i} = 1 - \frac{1}{2^n}$, and thus $\lim_{n\to+\infty} s_n = \lim_{n\to+\infty} 1 - \frac{1}{2^n} = 1 - 0 = 1$.

*Remark* 4.37. Notice that if the terms of a series are non-negative, then the sequence of partial sums is monotone increasing. Thus a series of positive terms either converges, or goes to infinity.

**Example 4.38.** The series $\sum_{n=1}^{\infty}(-1)^n$ has a sequence of partial sums $(-1, 0, -1, 0, \dots)$ and thus neither converges nor goes to infinity. But the terms are not all non-negative.

### 4.2.1 Geometric Series

Several of these examples are examples of an important type of series, called a geometric series.

**Definition 4.39.** A *geometric series* is a series $\sum_{n=1}^{\infty} ar^{n-1}$ for some real numbers $a$ and $r$.

Some people prefer to think of a geometric series as $\sum_{n=1}^{\infty} ar^n$. I'm one of them, actually, but Stewart isn't. It doesn't really matter which convention you use as long as you're consistent.

We'd like to figure out when geometric series converge, and to what. Since they have a regular pattern, we can do arithmetic to them. So let's start out assuming that $\sum_{n=1}^{\infty} ar^{n-1}$ converges to some number $L$. Then we have

$$\begin{aligned} rL &= \sum_{n=1}^{\infty} ar^n = \sum_{n=2}^{\infty} ar^{n-1} = L - a \\ (r-1)L &= -a \\ L &= \frac{a}{1-r}. \end{aligned}$$

Taken literally and doing no additional work, this suggests that $\sum_{n=1}^{\infty} 2^{n-1} = \frac{1}{1-2} = -1$, which is clearly absurd. (Well, usually. There's a trick called "regularization" that physicists use this for). But with a little more care we can work this out correctly.

Let $s_n = \sum_{i=1}^{n} ar^{n-1}$. Then

$$\begin{aligned} rs_n &= \sum_{i=1}^{n} ar^i = s_n - a + ar^n \\ (r-1)s_n &= a(r^n - 1) \\ s_n &= a\frac{r^n - 1}{r-1} \end{aligned}$$

If we take the limit as $n$ goes to infinity, this diverges if $|r| > 1$ and converges if $|r| < 1$; in that case, we have $\lim_{n\to+\infty} s_n = \frac{a}{1-r}$. Thus we have

**Proposition 4.40.** *If $\sum_{n=1}^{\infty} ar^{n-1}$ is a geometric series and $|r| < 1$, then*

$$\sum_{n=1}^{\infty} ar^{n-1} = \frac{a}{1-r}.$$

*If $|r \geq 1|$ then the series diverges.*

**Example 4.41.** We already argued that $\sum_{n=1}^{\infty} \frac{1}{2^n} = 1$. This is a geometric series with $a = r = \frac{1}{2}$.

**Example 4.42.** What is $\sum_{n=1}^{\infty} \frac{2}{3^n}$? We have $a = \frac{2}{3}, r = \frac{1}{3}$, so

$$\sum_{n=1}^{\infty} \frac{2}{3^n} = \frac{2/3}{1-1/3} = 1.$$

We can also use this technique to turn infinite repeating decimals into integer fractions.

**Example 4.43.** Can we write $4.\overline{13}$ as a ratio of integers?

We have

$$4.\overline{13} = 4 + \frac{13}{100} + \frac{13}{100^2} + \frac{13}{100^3} + \dots$$

After the first term we have a geometric series with $a = \frac{13}{100}$ and $r = \frac{1}{100}$, so the sum is

$$\frac{a}{1-r} = \frac{13/100}{99/100} = \frac{13}{99}.$$

Thus

$$4.\overline{13} = 4 + \frac{13}{99} = \frac{409}{99}.$$

**Example 4.44.** Does $\sum_{n=1}^{\infty} 3^{2n}2^{2-3n}$ converge or diverge?

This series looks like $\frac{9}{2} + \frac{3^4}{2^4} + \frac{3^6}{2^7} + \dots$. This is a geometric series with $a = \frac{9}{2}$ and $r = \frac{9}{8}$. Thus $|r| > 1$ and so the series diverges.

### 4.2.2 Other Easy Series Questions

Not all series are geometric. This is really unfortunate, since geometric series are so easy to deal with. There's another kind of series that's easy to compute.

**Example 4.45.** What is $\sum_{n=2}^{\infty} \frac{1}{n^2-n}$?

This isn't a geometric series. But we can go back to our definition of a series, and try to write down the sequence of partial sums. Our sequence looks like

$$\frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \dots$$

which looks like it converges. By doing a partial fraction decomposition, we can write $\frac{1}{n^2-n} = \frac{1}{n-1} - \frac{1}{n}$. Then our partial sums are

$$\begin{aligned} s_n &= \sum_{i=2}^{n} \frac{1}{i-1} - \frac{1}{i} \\ &= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \dots \\ &= 1 - \frac{1}{n}. \end{aligned}$$

Thus $\sum_{i=2}^{n} \frac{1}{n^2-n} = \lim_{n\to+\infty} 1 - \frac{1}{n} = 1$.

A series that works like this is called a *telescoping series*.

**Example 4.46.** Consider the series $\sum_{n=1}^{\infty} \log\left(\frac{n+1}{n}\right)$. We can look at this as

$$\sum_{n=1}^{\infty} \log(n+1) - \log(n).$$

Then we can observe

$$\begin{aligned} s_k &= \sum_{n=1}^{k} \log(n+1) - \log(n) \\ &= \big(\log(k+1) - \log(k)\big) + \big(\log(k) - \log(k-1)\big) + \dots + \big(\log(3) - \log(2)\big) + \big(\log(2) - \log(1)\big) \\ &= \log(k+1) - \log(1). \end{aligned}$$

$\lim_{k\to\infty} s_k = \lim_{k\to\infty} \log(k+1) = \infty.$

Thus this sum diverges.

### 4.2.3 Series Rules

**Proposition 4.47.** *If* $\sum a_n$ *and* $\sum b_n$ *are convergent series, then*

- $\sum ca_n = c \sum a_n$.
- $\sum (a_n \pm b_n) = \sum a_n \pm \sum b_n$.

*Remark* 4.48. There is a similar statement about $\left(\sum a_n\right)\left(\sum b_n\right)$:

$$\left(\sum_{n=0}^{\infty} a_n\right)\left(\sum_{n=0}^{\infty} b_n\right) = \sum_{n=0}^{\infty} \sum_{k=0}^{n} a_k b_{n-k}$$

This operation is sometimes referred to as *convolution*. It it is too complicated to be terribly useful to us right now, but it often comes up in signal processing and more sophisticated approaches to differential equations.

## 4.3 The Harmonic Series and the Divergence Test

Now we can talk about probably the most important single example of a series.

**Example 4.49.** One of the most important series is the *harmonic series* $\sum_{n=1}^{\infty} \frac{1}{n}$. (It underlies among other things the Riemann zeta function which controls the distribution of prime numbers). Does it converge or diverge?

There's no really generalizable argument that applies here. But if $s_n = \sum_{i=1}^{n} \frac{1}{n}$ is the sequence of partial sums, then

$$\begin{aligned} s_1 &= 1 > \frac{1}{2} \\ s_2 &= 1 + \frac{1}{2} > 2 \cdot \frac{1}{2} \\ s_4 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) > 3 \cdot \frac{1}{2} \\ s_8 &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8}\right) > 4 \cdot \frac{1}{2}. \end{aligned}$$

In particular, we see that $s_{2^{n-1}} > \frac{n}{2}$, and thus the sequence of partial sums increases without bound, and diverges to $+\infty$.

*Remark* 4.50. We will see that in some sense, the harmonic series is as small as it can get and still diverge.

**Example 4.51.** The Kempner Series is the harmonic series, except we leave out every term where a 9 appears in the denominator. We claim that this series converges. (Yes, seriously. See also `http://www.smbc-comics.com/index.php?id=3777`).

We divide the series up according to the number of digits in the denominator. Among denominators with $k$ digits, there are at most $8 \cdot 9^{k-1}$ since there are eight possibilities for the first digit (which cannot be 0 or 9) and 9 possibilities for the other digits (which cannot be 9). And each number is at least $10^{k-1}$, so each term with $k$ digits in the denominator is at most $10^{1-k}$.

Then if we sum up all the terms with $k$ digits in the denominator, we have $8 \cdot 9^{k-1}$ terms each of which is at most $10^{1-k}$ and so our sum is at most $\frac{8 \cdot 9^{k-1}}{10^{k-1}}$.

Now if we sum up the whole series, that's the same as summing up each set of $k$-digit denominators, and then summing all those sums. So we have

$$K \leq \sum_{k=1}^{\infty} 8 \frac{9^{k-1}}{10^{k-1}} = \sum_{k=1}^{\infty} 8 \left(\frac{9}{10}\right)^{k-1}.$$

This right-hand sum should look familiar; it's a geometric series. We have $a = 8$ and $r = \frac{9}{10}$, so the sum is

$$K \leq \sum_{k=1}^{\infty} 8 \left(\frac{9}{10}\right)^{k-1} = \frac{8}{1 - 9/10} = 80.$$

(A.J. Kempner first studied this series in 1914, and came up with the above argument. In 1979 Robert Baille showed that $K \approx 22.9$.)

*Remark* 4.52. In fact, if you take the harmonic series pick *any* string of digits, and remove terms with that string in the denominator, you get a convergent series, for basically the same reason.

Notice that we showed the harmonic series diverged by showing it was bigger than an infinite sum of a constant. This makes sense, because if you add the same number to itself infinitely many times, you will never get a finite amount. In fact, series can only converge if the terms get increasingly small as you go further into the series.

**Proposition 4.53.** *If* $\sum_{n=1}^{\infty} a_n$ *converges, then* $\lim_{n\to+\infty} a_n = 0$. *Thus if* $\lim_{n\to+\infty} a_n \neq 0$, *or if the limit does not exist, then* $\sum_{n=1}^{\infty} a_n$ *does not converge.*

*Remark* 4.54. The converse is not true. The divergence test can be used to show a series diverges; it cannot show that a series converges.

The divergence test winds up being a sort of first-pass filter. It lets us check that a series diverges really quickly, but can never tell us that a series converges.

**Example 4.55.** Consider the series $\sum_{n=1}^{\infty} 1$. We can see that $\lim_{n\to+\infty} 1 = 1 \neq 0$, so this series diverges. (We can see that in other ways by seeing that it must go to $\infty$.)

**Example 4.56.** Consider the series $\sum_{n=1}^{\infty} \frac{n}{n+1}$. We can see that $\lim_{n\to+\infty} \frac{n}{n+1} = 1 \neq 0$. Thus this series diverges.

**Example 4.57.** The divergence test tells us nothing about the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$. $\lim_{n\to+\infty} \frac{1}{n} = 0$, so we have no information. But we know that the harmonic series diverges by the argument in example 4.49.

This is a good example of how the divergence test can't show us a series converges. The harmonic series "passes" the divergence test: the terms go to zero. But that doesn't mean the series converges, and in fact it does not.

## 4.4 The Integral Test

**Example 4.58.** Let's look at the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. In lab last week we explored this series experimentally; we saw that $\sum_{i=1}^{10} \frac{1}{i^2} \approx 1.55$ and $\sum_{i=1}^{1000} \frac{1}{i^2} \approx 1.64$. The series appears to be converging somewhere, at least. Can we prove it converges?

Well, let's draw a picture. Let $f(x) = \frac{1}{x^2}$, and then the values of the sequence we're adding up are the points $f(n)$. Treat each of these points as the right endpoint of a rectangle of width one; then we see the integral of $f$ from 1 to $k$ is definitely larger than $\sum_{n=2}^{k} \frac{1}{n^2}$. (We leave out the first term of the series, but since that's a finite number it doesn't affect convergence). Thus

$$\sum_{n=2}^{k} \frac{1}{n^2} \leq \int_1^k \frac{1}{x^2}\, dx = \frac{-1}{x}\Big|_1^k = 1 - \frac{1}{k}.$$

Taking the limit gives a right hand side of 1, and thus the sum $\sum_{n=2}^{\infty} \frac{1}{n^2}$ is bounded and so must converge.

*Remark* 4.59. It turns out that the exact sum of this series is $\pi^2/6$. This was first proven by Leonhard Euler in 1734, originally establishing his reputation. The proof is moderately complicated and requires a number of tools relating to power series, which we will discuss later in the course. (If you're interested, look up the "Basel Problem" on Wikipedia).

**Example 4.60.** Does the series $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ converge?

We can use the same rough process and roughly the same picture we just did. By taking rectangles with *left* endpoints, we have

$$\sum_{n=1}^{k} \frac{1}{\sqrt{n}} \geq \int_1^k \frac{1}{\sqrt{x}}\, dx = 2\sqrt{x}|_1^k = \sqrt{k} - 1.$$

Taking the limit of both sides shows that $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} \geq \infty - 1$, and thus increases without bound.

We can build these types of argument into a general rule:

**Proposition 4.61** (Integral Test)**.** *Suppose $f$ is a continuous, positive, decreasing function on $[m, +\infty)$ for some $m$. Let $a_n = f(n)$. Then the series $\sum_{n=m}^{\infty} a_n$ converges if and only if $\int_m^{+\infty} f(x)\, dx$ converges. That is:*

- *If $\int_m^{\infty} f(x)\, dx$ converges then $\sum_{n=m}^{\infty} a_n$ converges.*
- *If $\int_m^{\infty} f(x)\, dx$ diverges then $\sum_{n=m}^{\infty} a_n$ diverges.*

*Remark* 4.62. Note that this doesn't tell us what the sum of the series is, just that it exists. In general, if we want to know the exact sum of a series we need a way to write a closed-form formula for the sequence of partial sums; most of the rest of the tools we'll develop in this class will only be used to establish that some series converges at all.

**Example 4.63.** Does $\sum_{n=1}^{\infty} \frac{2n}{n^2+1}$ converge?

Let $f(x) = \frac{2x}{x^2+1}$. Then $f$ is clearly positive and continuous, and $f'(x) = \frac{2(x^2+1)-4x^2}{(x^2+1)^2}$ is negative so $f$ is decreasing. So we can use the integral test.

$$\begin{aligned}\int_1^{+\infty} f(x)\,dx &= \lim_{t\to+\infty} \int_1^t \frac{2x}{x^2+1}\,dx \\ &= \lim_{t\to+\infty} \ln|x^2+1||_1^t = \lim_{t\to+\infty} \ln|t^2+1| - \ln|2| = +\infty.\end{aligned}$$

So $\int_1^{+\infty} f(x)\,dx$ diverges, and thus so does $\sum_{n=1}^{\infty} \frac{2n}{n^2+1}$.

**Proposition 4.64.** *The series* $\sum_{n=1}^{\infty} \frac{1}{n^p}$ *converges if* $p > 1$ *and diverges if* $p \leq 1$.

*Proof.* If $p = 1$ this is the harmonic series, and we know it diverges.

If $p \neq 1$ then $f(x) = \frac{1}{x^p}$ is a positive, decreasing, continuous function, so we can use the integral test. We have

$$\int_1^{+\infty} \frac{1}{x^p}\,dx = \lim_{t\to+\infty} \int_1^t x^{-p}\,dx = \lim_{t\to+\infty} \frac{x^{1-p}}{1-p}|_1^t = \lim_{t\to+\infty} \frac{t^{1-p}}{1-p} - \frac{1}{1-p}.$$

This converges precisely when $1 - p < 0$, precisely when $p > 1$. ☐

**Example 4.65.** Does $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ converge or diverge?

We again see that $f(x) = \frac{1}{x^2+1}$ is a positive, decreasing, continuous function, so we can use the integral test. We have

$$\int_1^{+\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to+\infty} \int_1^t \frac{1}{x^2+1}\,dx = \lim_{t\to+\infty} \arctan(x)|_1^t = \lim_{t\to+\infty} \arctan t - \arctan 1 = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

So the series converges by the integral test.

But there's also an easier way to see this converges. We know that $\frac{1}{n^2+1} \leq \frac{1}{n^2}$, so intuitively if $\sum \frac{1}{n^2}$ is finite then $\sum \frac{1}{n^2+1}$ should be as well. This leads us to:

## 4.5 The Comparison Test

The integral test is powerful, and you can in theory answer nearly any question about positive series with the divergence test and the integral test combined. But in practice, the integral test can be really annoying to use, since we have to actually compute integrals. We want to use the work we've already done to avoid having to do more work.

We can do that by comparing new series to old series we've already worked out.

**Proposition 4.66** (Comparison Test). *Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are series with positive terms. Then:*

- *If $\sum_{n=1}^{\infty} a_n$ converges and $a_n \geq b_n$ for all (sufficiently large) $n$, then $\sum_{n=1}^{\infty} b_n$ converges.*
- *If $\sum_{n=1}^{\infty} a_n$ diverges and $a_n \leq b_n$ for all (sufficiently large) $n$, then $\sum_{n=1}^{\infty} b_n$ diverges.*

*Remark* 4.67. Note that this only applies to series with positive terms. $\sum \frac{1}{2^n}$ converges and $-1 \leq \frac{1}{2^n}$ for all $n$, but $\sum_{n=1}^{\infty}(-1)$ does not converge.

*Remark* 4.68. Using the comparison test requires us to have something to compare our series with. We usually use a power series $\sum n^p$ or a geometric series $\sum ar^{n-1}$.

**Example 4.69.** Does $\sum_{n=1}^{\infty} \frac{1}{n^3+n^2+n+1}$ converge?

We know that $n^3 \leq n^3+n^2+n+1$, so $\frac{1}{n^3+n^2+n+1} \leq \frac{1}{n^3}$. Since $\sum \frac{1}{n^3}$ converges, we know that $\sum_{n=1}^{\infty} \frac{1}{n^3+n^3+n+1}$ converges by the Comparison Test.

**Example 4.70.** Does $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ converge?

We could use the integral test, but we can also comment that $\ln n \geq 1$ for $n \geq 3$, so $\frac{\ln n}{n} \geq \frac{1}{n}$. Since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, we know that $\sum_{n=1}^{\infty} \frac{\ln n}{n}$ diverges by the comparison test.

**Example 4.71.** Does $\sum_{n=1}^{\infty} \frac{1}{n!}$ converge or diverge?

For $n > 2$, we know that $n! > n^2$, so $\frac{1}{n!} \leq \frac{1}{n^2}$. Thus the series $\sum_{n=1}^{\infty} \frac{1}{n!}$ converges by the comparison test.

Alternatively: $n! > 2^{n-1}$, and thus $\frac{1}{n!} \leq \frac{1}{2^{n-1}}$. But $\sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$ is a geometric series and converges since $r = 1/2 < 1$, so by the comparison test $\sum_{n=1}^{\infty} \frac{1}{n!}$ also converges.

**Example 4.72.** Does $\sum_{n=1}^{\infty} \frac{1}{n^3-n^2+1}$ converge?

This is a lot harder, because it's not actually true that $\frac{1}{n^3-n^2+1} \leq \frac{1}{n^3}$ (in fact $n^3 > n^3 - n^2 + 1$ for $n > 1$).

We can try to fix this by making the simple comparison bigger; instead of $1/n^3$ maybe we'll try $2/n^3$. And it turns out to be true that $n^3/2 < n^3 - n^2 + 1$ for $n > 1$, since $n^2 < n^3/2 + 1$. So $\frac{1}{n^3-n^2+1} \leq \frac{2}{n^3}$. This is enough to show that $\sum_{n=1}^{\infty} \frac{1}{n^3-n^2+1}$ converges by the comparison test.

This argument works, but it's fiddly and annoying and seems like it must be too complicated; we'd like to be able to say that $\frac{1}{n^3-n^2+1}$ looks "basically like" $\frac{1}{n^3}$ and so they behave the same. Fortunately there's a way to make that work out.

**Proposition 4.73** (Limit Comparison Test)**.** *Suppose $\sum a_n$ and $\sum b_n$ are series with positive terms, and $\lim_{n\to+\infty} \frac{a_n}{b_n}$ exists and is a finite, nonzero number. Then either both series converge, or both series diverge.*

Thus we have

$$\lim_{n\to+\infty} \frac{1/n^3}{1/(n^3-n^2+1)} = \lim_{n\to+\infty} \frac{n^3-n^2+1}{n^3} = 1,$$

and since $\sum_{n=1}^{\infty} \frac{1}{n^3}$ converges, so does $\sum_{n=1}^{\infty} \frac{1}{n^3-n^2+1}$.

**Example 4.74.** Does $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+n^2+1}$ converge?

We suspect we can compare this to $\frac{n^2}{n^3}$, or in fact to $\frac{1}{n}$, which has matching top degree. We check by calculating

$$\lim_{n\to+\infty} \frac{\frac{n^2+1}{n^3+n^2+1}}{1/n} = \lim_{n\to+\infty} \frac{n^3+n}{n^3+n^2+1} = \lim_{n\to+\infty} \frac{1+n^{-2}}{1+n^{-1}+n^{-3}} = 1.$$

This is a real number between 0 and $+\infty$. Thus, since $\sum_{n=1}^{\infty} \frac{1}{n}$ does not converge, by the limit comparison test $\sum_{n=1}^{\infty} \frac{n^2+1}{n^3+n^2+1}$ also diverges.

**Example 4.75.** Does $\sum_{n=1}^{\infty} \frac{n+5}{\sqrt{n^5+n^3+n}}$ converge or diverge?

The numerator has the order of $n$ and the denominator has the order of $n^{5/2}$, so we want to compare this to $\frac{n}{n^{5/2}} = \frac{1}{n^{3/2}}$. So we calculate

$$\begin{aligned} \lim_{n\to\infty} \frac{\frac{n+5}{\sqrt{n^5+n^3+n}}}{1/n^{3/2}} &= \lim_{n\to\infty} \frac{n^{5/2}+5n^{3/2}}{\sqrt{n^5+n^3+n}} \\ &= \lim_{n\to\infty} \frac{1+5/n}{\sqrt{1+1/n^2+1/n^4}} = 1. \end{aligned}$$

This is a real number in $(0,\infty)$, and thus the two series have the same convergence behavior.

Since $3/2 > 1$, by the $p$-series test we know that $\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ converges. So by the limit comparison test, $\sum_{n=1}^{\infty} \frac{n+5}{\sqrt{n^5+n^3+n}}$ converges.

**Example 4.76.** Does the series $\sum_{n=1}^{\infty} \frac{1}{3^n-2}$ converge or diverge?

We can't really use the regular comparison test here; the obvious point of comparison is $\sum \frac{1}{3^n}$, but $\frac{1}{3^n-2} > \frac{1}{3^n}$. But we can compute

$$\lim_{n\to\infty} \frac{1/(3^n-2)}{1/3^n} = \lim_{n\to\infty} \frac{3^n}{3^n-2} = \lim_{n\to\infty} \frac{1}{1-2/3^n} = 1.$$

Thus by the limit comparison test, $\sum_{n=1}^{\infty} \frac{1}{3^n-2}$ converges.

We *could* talk a lot more about limit comparison to a geometric series, but there'll be a better way to handle this in section 4.7 when we talk about the ratio test.

## 4.6 Non-Positive Series

So far we've only discussed series with all positive terms, and we have a pretty good handle on them: we use the integral test to work out some basic examples, and then solve others with the comparison tests.

Things get a little trickier when we want to talk about series that include negative terms. They can get very complicated, but we'll start off with an easy type of example.

### 4.6.1 Alternating Series

**Definition 4.77.** An *alternating series* is a series whose terms are alternately positive and negative: either all the odd terms are negative and the even terms are positive, or all the even terms are negative and all the odd terms are positive.

**Example 4.78.** Some alternating series are

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots$$
$$\sum_{n=1}^{\infty} (-1)^n \frac{n^2}{n+3} = -\frac{1}{4} + \frac{4}{5} - \frac{9}{6} + \frac{16}{7} - \dots$$

Every alternating series $\sum a_n$ looks like $\sum (-1)^n |a_n|$ or $\sum (-1)^{n-1} |a_n|$.

**Proposition 4.79** (Alternating Series Test)**.** *If* $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ *is an alternating series such that* $b_{n+1} < b_n$ *for all (sufficiently large)* $n$*, and* $\lim_{n \to +\infty} b_n = 0$*, then the series is convergent.*

*Sketch of Proof.* $\sum_{n=1}^{\infty} (-1)^{n-1} b_n = b_1 - (b_2 - b_3) - (b_4 - b_5) - \dots \leq b_1$ is bounded above, and $(b_1 - b_2) + (b_3 - b_4) + \dots$ is increasing, so the sequence of even partial sums must converge. It's not hard to see that the sequence of odd partial sums must have the same limit. ☐

**Example 4.80.** The alternating harmonic series $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \dots = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges by the alternating series test, since $\frac{1}{n+1} < \frac{1}{n}$ and $\lim_{n \to +\infty} \frac{1}{n} = 0$.

**Example 4.81.** The series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n+1}$ does not converge. The series is alternating, but the alternating series test does not apply because $\lim_{n \to +\infty} \frac{n}{n+1} = 1 \neq 0$. In fact, we see that $\lim_{n \to +\infty} (-1)^{n-1} \frac{n}{n+1}$ does not exist, so by the divergence test this series diverges.

**Example 4.82.** The series $\sum_{n=1}^{\infty} (-1)^n \frac{n^3}{n^4+2}$ converges. The sequence $\frac{n^3}{n^4+2}$ is decreasing, as we can see by taking the derivative of $f(x) = \frac{x^3}{x^4+2}$. Further, the limit is zero, so by the alternating series test the series converges.

We'll note that our results taken together mean we can test the convergence of any alternating series really easily. If the terms go to zero, it converges by the alternating series test; if the terms don't go to zero, it diverges by the divergence test.

Thus normally the divergence test is a necessary but not sufficient condition. For an alternating series specifically, it is both necessary and sufficient.

One other nice thing about alternating series is that we have a very good estimate of how close we are to the true sum. That means we can calulate estimates fairly easily, and know exactly how many terms we need to work out to be correct within our desired margin of error.

**Proposition 4.83** (Alternating Series Estimation). *If $s = \sum_{n=1}^{\infty} (-1)^{n-1} b_n$ is an alternating series that satisfies the hypotheses of the Alternating Series Test, then $|s - s_n| \leq b_{n+1}$.*

**Example 4.84.** Consider the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$. What is the error term in approximating the sum if we calculate the first ten terms?

The size of the error is smaller than the next term, which is the eleventh term, which is $\frac{1}{121}$. Thus $\sum_{n=1}^{10} \frac{(-1)^n}{n^2}$ is within $\frac{1}{121}$ of the infinite sum (which we will see is $\log 2 \approx .7$).

**Example 4.85.** Consider the alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2+2n+1}$. How many terms do we have to calculate to get the answer to within 1/100?

The ninth term has size $\frac{1}{9^2+18+1} = \frac{1}{100}$, so we need to compute the first eight terms.

### 4.6.2 Absolute Convergence

The alternating series test allowed us to study one particular type of series with non-positive terms, but alternating series are just one way a series can have non-positive terms. The idea of absolute convergence allows us to handle many more types of series.

**Definition 4.86.** A series $\sum a_n$ is called *absolutely convergent* if $\sum |a_n|$ converges.

A series $\sum a_n$ is *conditionally convergent* if it is convergent but not absolutely convergent.

**Theorem 4.87.** *If $\sum a_n$ is absolutely convergent, then it converges.*

*Proof.* $0 \leq a_n + |a_n| \leq 2|a_n|$, and $a_n + |a_n| \geq 0$. We have $\sum_{n=1}^{\infty} 2|a_n|$ converges, so by comparision test $\sum_{n=1}^{\infty} a_n + |a_n|$ converges. But then

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (a_n + |a_n|) - \sum_{n=1}^{\infty} a_n$$

is a difference of convergent series and so converges. ☐

*Remark* 4.88. The converse is not true! $\sum_{n=1}^{\infty}(-1)^n/n$ is convergent (by the alternating series test) but not absolutely convergent. This is why it's possible to be conditionally convergent.

This theorem lets us study many sequences with positive and negative terms.

**Example 4.89.** The series $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ is absolutely convergent. We have $\left|\frac{\sin n}{n^2}\right| \leq \frac{1}{n^2}$ and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, so by the comparison test $\sum \left|\frac{\sin n}{n^2}\right|$ converges.

**Example 4.90.** The alternating series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ converges by the alternating series test, but $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n}\right| = \sum_{n=1}^{\infty} \frac{1}{n}$ diverges. So the series is conditionally convergent.

**Example 4.91.** The series $\sum_{n=1}^{\infty} \sin n$ diverges by the divergence test, since $\lim_{n\to+\infty} \sin n$ does not exist.

**Example 4.92.** We claim that $\sum_{n=1}^{\infty}(-1)^n/n^2$ converges absolutely. For $\sum_{n=1}^{\infty} |(-1)^n/n^2| = \sum_{n=1}^{\infty} n^{-2}$ which we know converges.

The main purpose of this is to take questions about series with complex terms, and turn them into questions about series with positive real terms, so that our previous tests apply.

**Proposition 4.93.** *If a series is absolutely convergent, then the sum doesn't depend on the order of the terms. (In particular, the sum of a series of positive numbers doesn't depend on the order of the terms).*

*If a series is conditionally convergent but not absolutely convergent, then the sum* does *depend on the order of the terms; and in fact by reordering the terms we can get essentially any sum we like.*

*More precisely, if $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent real series, then by reordering we can cause the sum to converge to any real number, or to diverge to $+\infty$ or $-\infty$.*

**Example 4.94.** It's possible to compute that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln 2$. But we also have

$$1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \cdots = \frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \cdots = \frac{\ln 2}{2}.$$

*Proof.* Suppose $\sum_{n=1}^{\infty} a_n$ is a conditionally convergent series of real numbers. Rewrite it as $\sum_{n=1}^{\infty} b_n - \sum_{n=1}^{\infty} c_n$ where the $b_n$ are all the positive terms and the $c_n$ are all the negative terms. If both of these sums converged, then the series would converge absolutely (since $\sum b_n + \sum c_n = \sum b_n + c_n = \sum |a_n|$); if one converged and the other diverged, then $\sum a_n$ would diverge. So $\sum b_n = \sum c_n = +\infty$.

Pick a target $M$. Arrange the sum as follows: include positive terms until the sum is above $M$. Then include negative terms until the sum is below $M$. Repeat, alternating, infinitely. The sum will oscillate around $M$ and converge to $M$.

If we want the sum to approach $+\infty$, include positive terms until the sum is above 1, then a negative term, then positive terms until the sum is above 2, then a negative term, and so on. $\square$

## 4.7 The Ratio Test

Once we know to look for absolute convergence, we can use the comparison test on any series, but we'd like to cut out some steps. If we imagine comparing our series to a geometric series, we get the *ratio test*:

**Proposition 4.95** (Ratio Test). *If $\sum_{n=1}^{\infty} a_n$ is a series and $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, then:*

- *If $L < 1$ then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.*
- *If $L > 1$ then the series $\sum_{n=1}^{\infty} a_n$ diverges.*

*Remark* 4.96. If $\lim_{n\to+\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$ or does not exist, then the ratio test tells us nothing. We have to use some other test or technique.

**Lemma 4.97.**

$$\lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^n = e.$$

**Example 4.98.** Analyze the convergence of $\sum_{n=1}^{\infty} \frac{1}{n!}$.

Since it has a factorial, this is a natural place to apply the ratio test. We have

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{1/(n+1)!}{1/n!}\right| = \lim \frac{1}{n+1} = 0 < 1,$$

so by the ratio test this series converges absolutely.

**Example 4.99.** Analyze the convergence of $\sum_{n=1}^{\infty} \frac{n!}{n^n}$.

Again, there are factorials so we want to use the ratio test. We have

$$\lim \left|\frac{a_{n+1}}{a_n}\right| = \lim \left|\frac{(n+1)!/(n+1)^{n+1}}{n!/n^n}\right| = \lim \frac{(n+1)n^n}{(n+1)^{n+1}} = \lim \left(\frac{n}{n+1}\right)^n = \frac{1}{e} < 1$$

so by the ratio test this series converges absolutely.

**Example 4.100.** What about $\sum_{n=1}^{\infty} \frac{r^n}{n!}$? For what $r$ does it converge?

We still want to use the ratio test. We have

$$\lim \left|\frac{r^{n+1}/(n+1)!}{r^n/n!}\right| = \lim \frac{r}{n+1} = 0 < 1.$$

By the ratio test, this converges absolutely for any $r$.

**Example 4.101.** Now let $r > 0$ be a real number. Does $\sum_{n=1}^{\infty} \frac{n!}{r^n}$ converge or diverge?

This is similar but opposite to the previous problem. We have

$$\lim \left| \frac{(n+1)!/r^{n+1}}{n!/r^n} \right| = \lim \frac{n+1}{r} = +\infty > 1$$

so by the ratio test this diverges.

**Example 4.102.** Analyze the convergence of $\sum_{n=1}^{\infty} \frac{n^2+1}{2^n}$.

We compute

$$\lim \left| \frac{((n+1)^2+1)/(2^{n+1})}{(n^2+1)(2^n)} \right| = \lim \frac{n^2+2n+1}{(n^2+1)\cdot 2} = \frac{1}{2} < 1.$$

So by the ratio test this converges.

## 4.8 The Root Test

The Root Test is similar to the ratio test, but is sometimes slightly easier or harder to apply than the Ratio Test is.

**Proposition 4.103** (Root Test). *If $\sum_{n=1}^{\infty} a_n$ is a series and $\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$, then:*

- *If $L < 1$ then the series $\sum_{n=1}^{\infty} a_n$ converges absolutely.*
- *If $L > 1$ then the series $\sum_{n=1}^{\infty} a_n$ diverges.*

*Remark* 4.104. If $\lim_{n\to+\infty} \sqrt[n]{|a_n|} = 1$ or does not exist, then the root test tells us nothing. We have to use some other test or technique.

**Example 4.105.** Analyze $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n+1}{5n+2}\right)^n$.

We have $a_n = \left(\frac{2n+1}{5n+2}\right)^n$ and thus $\sqrt[n]{|a_n|} = \frac{2n+2}{5n+2}$. So $\lim_{n\to+\infty} \sqrt[n]{|a_n|} = \frac{2}{5} < 1$ so the series converges absolutely.

**Example 4.106.** Analyze $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2n^2+1}{3n^2+2n+1}\right)^n$

Our terms have a $n$th power in them, so the root test seems natural. We compute

$$\lim \sqrt[n]{\left| \frac{2n^2+1}{3n^2+2n+1} \right|} = \lim \frac{2n^2+1}{3n^2+2n+1} = \frac{2}{3} < 1.$$

So by the Root Test this series converges absolutely.