5 Power Series and Taylor Series

In this section we want to use what we've done with series in order to accomplsih things. It turns out that we can build functions out of series. And once we do this, a lot of the calculus we've done before becomes much simpler.

5.1 Power Series

We want to start by figuring out how to build a function out of a series. There are a few ways to do this, but the one we'll be studying is called a *power series*.

(I might talk about Fourier series briefly at the end of the course, depending on how things go.)

Definition 5.1. A *power series* is a series

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots$$

Note we assume that $x^0 = 1$. More generally, a *power series centered at a* is a series

$$\sum_{n=1}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \dots$$

We've seen a couple of these in disguise before.

Example 5.2. The series

$$1 + x + x^2 + x^3 + \cdots = \sum_{n=0}^{\infty} x^n$$

is a geometric series with $a = 1$ and $r = x$. We know from our study of geometric series that this converges if $|x| < 1$ and diverges if $|x| \geq 1$, and that when it converges, the sum is $\frac{1}{1-x}$.

Example 5.3. For what x does $\sum_{n=0}^{\infty} n(x-2)^n$ converge?

We use the ratio test (as we often do for power series). We have $a_n = n(x-2)^n$ and $a_{n+1} = (n+1)(x-2)^{n+1}$, so

$$\lim_{n\to+\infty} \left| \frac{(n+1)(x-2)^{n+1}}{n(x-2)^n} \right| = \lim_{n\to+\infty} \left| \frac{n+1}{n}(x-2) \right| = |x-2|$$

and this is < 1 precisely when $|x-2| < 1$. For real numbers this is when $1 < x < 3$.

When $|x-2| = 1$ we have to do a bit more work; it's not too hard to check for $x = 1$ or $x = 3$, but we do have to check them individually.

When $x = 3$ then our series is $\sum_{n=0}^{\infty} n \cdot 1^n$ which clearly diverges by the divergence test. Similarly, if $x = 1$ our series is $\sum_{n=0}^{\infty} n \cdot (-1)^n$ which again diverges by the divergence test.

Example 5.4. For what x does $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converge?

Again, we use the ratio test. (Honestly, we almost always use the ratio test). Since $a_n = \frac{x^n}{n!}$, we have

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{x^{n+1}/(n+1)!}{x^n/n!} \right| = \lim \left| \frac{x^{n+1} n!}{(n+1)! x^n} \right| = \lim \left| \frac{x}{n} \right| = 0.$$

So this series converges absolutely for any real number x.

Theorem 5.5. *If $\sum_{n=0}^{\infty} c_n (x-a)^n$ is a power series, then exactly one of the following things occurs:*

- *The series converges only when $x = a$;*
- *The series converges for any real number x;*
- *There is a positive number R, called the* radius of convergence*, such that the power series converges for $|x-a| < R$ and diverges for $|x-a| > R$. Note this tells us nothing about what happens when $|x-a| = R$; we have to check those cases individually.*

Remark 5.6. This explains the language of "absolute" and "conditional" convergence, which was developed for power series. A power series will converge everywhere on the interior of its interval of convergence. It diverges everywhere outside the interval. On the boundary of the interval it may or may not converge, depending on the specific boundary point; thus, on the boundary it converges "conditionally."

(This can all generalize to complex numbers in a really important and interesting way, but we're not going to engage with that much in this course. But in the complex case, you can replace the word "interval" with "disk" in this remark.)

Definition 5.7. The *open interval of radius r centered at c*, is

$$(c-r, c+r) = \{x : |x-c| < r\}$$

the set of all points of distance *less than* r from the center c.

The *closed interval* of radius r centered at c is

$$[c-r, c+r] = \{x : |x-c| \leq r\}$$

the set of all points of distance at most r from the center c.

Note the closed interval contains its boundary points and the open interval does not. This is important!

Example 5.8. For what real x does $\sum_{n=0}^{\infty} \frac{(x-4)^n}{n}$ converge? What is the radius of convergence?

Guess what? We use the ratio test! We have

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{(x-4)^{n+1}/(n+1)}{(x-4)^n/n} \right| = \lim \left| \frac{(x-4)n}{n+1} \right| = \lim |x-4|$$

converges for $|x-4| < 1$, and thus the radius of convergence is 1. The power series converges absolutely on $(3, 5)$.

To find the real numbers where the series converges, we have to check 3 and 5 manually. For $x = 3$ we get the series $\sum \frac{(-1)^n}{n}$ which converges by the Alternating Series Test; and for $x = 5$ we get the series $\sum \frac{1}{n}$ which diverges. Thus the series converges on $[3, 5)$ in the real numbers.

Example 5.9. The Bessel function (of order 0) is critical to any physics done in cylindrical coordinates, and thus any physics that occurs on a cylinder. It is defined by the power series:

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

What is the interval of convergence?

We use the ratio test. We have $a_n = \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$, so

$$\lim \left| \frac{a_{n+1}}{a_n} \right| = \lim \left| \frac{x^{2n+2}/2^{2n+2}((n+1)!)^2}{x^{2n}/2^{2n}(n!)^2} \right| = \lim \frac{|x|^2}{4(n+1)^2} = 0.$$

Thus the Bessel function of order 0 converges absolutely for all real numbers x. We say the radius of convergence is ∞ and the interval is all reals, or $(-\infty, +\infty)$.

Example 5.10. What is the radius and interval of convergence of

$$\sum_{n=1}^{\infty} \frac{(-2)^n x^n}{\sqrt{n^2+n}}?$$

Ratio test.

$$\lim \left| \frac{(-2)^{n+1} x^{n+1}/\sqrt{(n+1)^2+n+1}}{(-2)^n x^n/\sqrt{n^2+n}} \right| = \lim 2|x| \frac{\sqrt{n^2+3n+2}}{\sqrt{n^2+n}} = 2|x|.$$

Thus by the ratio test the power series converges absolutely when $|x| < 1/2$ so the radius of convergence is $1/2$ and it converges on the open interval $(-1/2, 1/2)$. Now we need to test endpoints.

When $x = 1/2$ then the series is $\sum \frac{(-1)^n}{\sqrt{n^2+n}}$. This is an alternating series; the terms are decreasing and tend towards zero, so it converges. When $x = -1/2$ then the series is $\sum \frac{1}{\sqrt{n^2+n}}$. We use the limit comparison test, and see that

$$\lim \frac{1/n}{1/\sqrt{n^2+n}} = \lim \sqrt{1+1/n} = 1$$

and thus $\sum \frac{1}{\sqrt{n^2+n}}$ has the same behavior as $\sum \frac{1}{n}$, and thus diverges.

So the real interval of convergence is $(-1/2, 1/2]$.

Example 5.11. What is the interval of convergence of

$$\sum_{n=0}^{\infty} \frac{n^2(x-1)^n}{7^{n+2}}?$$

Using the ratio test, we have

$$\lim \left| \frac{(n+1)^2(x-1)^{n+1}/7^{n+3}}{n^2(x-1)^n/7^{n+2}} \right| = \lim \frac{|x-1|}{7} \frac{(n+1)^2}{n^2} = \frac{|x-1|}{7}.$$

So the series converges absolutely when $|x-1| < 7$, and thus on the interval $(-6, 8)$. For the full interval we need to test the endpoints, at $x = -6$ and $x = 8$.

When $x = -6$ we have $\sum \frac{n^2(-7)^n}{7^{n+2}} = \sum (-1)^n \frac{n^2}{49}$. This is an alternating series, but the terms tend towards infinity and so by the divergence test it diverges.

Similarly, when $x = 8$ we have $\sum \frac{n^2 7^n}{7^{n+2}} = \sum \frac{n^2}{49}$. The terms tend towards infinity, so the series diverges by the divergence test.

Thus the real interval of convergence is $(-6, 8)$.

Example 5.12. What are the radius and interval of convergence for $\sum_{n=0}^{\infty} \left(\frac{6n^2+5n+3}{3n^2+2n+1} \right)^n (x-3)^n$?

Since everything is nth powers, we use the root test. We have

$$\lim \sqrt[n]{\left| \left(\frac{6n^2+5n+3}{3n^2+2n+1} \right)^n (x-3)^n \right|} = \lim \frac{6n^2+5n+3}{3n^2+2n+1} |x-3| = 2|x-3|.$$

This is absolutely convergent when $|x-3| < 1/2$, so the radius of convergence is $1/2$ and it converges on the open interval $(5/2, 7/2)$.

When $x = 7/2$ then our series is

$$\sum \left(\frac{6n^2+5n+3}{3n^2+2n+1} \right)^n \frac{1}{2^n} = \sum \left(\frac{6n^2+5n+3}{6n^2+4n+2} \right)^n$$

and the terms do not converge to zero, so it diverges by the divergence test. Similarly if $x = 5/2$ then the series diverges by the divergence test.

Thus $\sum \left(\frac{6n^2+5n+3}{3n^2+2n+1} \right)^n (x-3)^n$ is absolutely convergent on $(5/2, 7/2)$ and divergent elsewhere on the real line.

5.2 Power Series as Functions

Now that we understand how power series converge, we can return to analyzing them as functions. In general, if we have a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, then we can define a function by $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$.

The domain of the function will be the interval of convergence of the power series. I.e. if $= \sum c_n(x-a)^n$ has radius of convergence $R > 0$ then f is defined on $(a-R, a+R)$, and depending on conditional convergence it may be defined on various boundary points as well. If the power series converges everywhere then the domain is all real numbers.

Recall we observed that $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $|x| < 1$. From this alone we can describe some functions in terms of power series.

Example 5.13. $\frac{1}{1-x^2} = \sum_{n=0}^{\infty}(x^2)^n$ and $\frac{1}{1+x^5} = \sum_{n=0}^{\infty}(-x^5)^n = \sum(-1)^n x^{5n}$. Note that both of these have the same interval of convergence as the original power series, since x^2 and $-x^5$ are in $(-1,1)$ precisely when x is. So the interval of convergence, and thus the domain of the function defined by the power series, is $(-1,1)$.

Remark 5.14. Note that $\frac{1}{1-x}$ has a domain bigger than $(-1,1)$. But the power series only converges on that smaller interval. (The power series converges on an interval centered at zero; what on the boundary that suggests it will stop converging there?)

Example 5.15. How can we express $\frac{1}{x-3}$ as a power series? Since we want to write the denominator as $1-y$ for some expression y, we factor out a -3:

$$\frac{1}{x-3} = \frac{1}{-3} \cdot \frac{1}{1-x/3} = -\frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{x}{3}\right)^n = \sum_{n=0}^{\infty} \frac{-1}{3^{n+1}} x^n.$$

We know this will converge when $\left|\frac{x}{3}\right| < 1$, and thus when $|x| < 3$. So the interval of convergence is $(-3,3)$.

We can also do most basic algebra with power series.

Example 5.16. How can we express $\frac{x}{1-x}$ as a power series? This is just $x \cdot \frac{1}{1-x}$ and so

$$\frac{x}{1-x} = x\sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} x^{n+1}.$$

The interval of convergence is again $(-1,1)$.

5.2.1 Calculus of power series

Recall that possibly the easiest functions for us to work with when we do calculus (or, indeed, almost anything else) are polynomials. It's easy to differentiate or integrate polynomials, and to calculate their outputs. The nice thing about power series is that they're basically fake polynomials, so they're almost as good:

Proposition 5.17. *If $\sum c_n(x-a)^n$ has a radius of convergence $R > 0$, then the function defined by $f(x) = \sum c_n(x-a)^n$ is differentiable on $(a-R, a+R)$, and we have*

- $f'(x) = \sum_{n=0}^\infty c_n \frac{d}{dx}((x-a)^n) = \sum_{n=0} nc_n(x-a)^{n-1}$.
- $\int f(x)\,dx = \sum_{n=0}^\infty \left(\int c_n(x-a)^n\,dx\right) = \sum_{n=0}^\infty c_n \frac{(x-a)^{n+1}}{n+1} + C$.

Remark 5.18. This proposition tells us that after taking the derivative or integral, our power series still has the same radius of convergence. However, convergence at the *endpoints* may change.

Now that we have this extra tool we can find power series for more functions.

Example 5.19. Since $\frac{1}{1-x} = \sum_{n=0}^\infty x^n$, we can differentiate both sides and get

$$\frac{1}{(1-x)^2} = \sum_{n=1}^\infty nx^{n-1}.$$

Note that we've dropped the $n = 0$ term because the derivative of x^0 is 0, and writing $0 \cdot x^{-1}$ would be silly.

Also note that we could write instead

$$\frac{1}{(1-x)^2} = \sum_{n=0}^\infty (n+1)x^n$$

so this is still a proper power series. We can use the ratio test to see that the radius of convergence is still $R = 1$.

Example 5.20. Can we find a power series expression for $\ln(1+x)$?

We know that $\ln(1+x) = \int \frac{1}{1+x}\,dx$. We also know that $\frac{1}{1+x} = \sum_{n=0}^\infty (-x)^n$. Integrating gives us that

$$\begin{aligned}
\ln(1+x) &= \sum_{n=0}^\infty \frac{(-x)^{n+1}}{n+1} \cdot (-1) + C \\
&= \sum_{n=0}^\infty (-1)^n \frac{x^{n+1}}{n+1} + C \\
&= \sum_{n=1}^\infty (-1)^{n-1} \frac{x^n}{n} + C.
\end{aligned}$$

To find the constant we compute $\ln(1+0)$, since plugging 0 in on the right hand side will just yield C. $\ln(1) = 0$ so $C = 0$, and we have

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

The radius of convergence is still 1; we can see this using the ratio test or by recalling that our original power series has radius of convergence 1.

After some thought, we see that the radius of convergence can't possibly be larger than 1, since $\ln(1+(-1))$ isn't well defined and the limit is infinity.

Remark 5.21. This proves my earlier claim that $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = \ln(2)$, by evaluating at $x = 1$.

Example 5.22. Find a power series for $\arctan x$.

Again, we note that $\arctan x = \int \frac{1}{1+x^2}\,dx$, and we know that $\frac{1}{1+x^2} = \sum_{n=0}^{\infty}(-x^2)^n$. Integrating gives

$$\begin{aligned}\arctan x &= \sum_{n=0}^{\infty} \int (-1)^n x^{2n}\,dx + C \\ &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} + C.\end{aligned}$$

To find C we calculate $C = \arctan 0 = 0$, so we have

$$\arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

Finally, these power series representations allow us to compute integrals that we either couldn't do or couldn't do easily before. We'll see more of this soon.

Example 5.23. What is $\int \frac{1}{1+x^6}$?

We could use a partial fractions decomposition, if we know that $1 + x^6 = (1+x^2)(x^2 - \sqrt{3}x + 1)(x^2 + \sqrt{3}x + 1)$, but that's really unpleasant. Instead, we write

$$\begin{aligned}\frac{1}{1+x^6} &= \sum_{n=0}^{\infty}(-x^6)^n = \sum_{n=0}^{\infty}(-1)^n x^{6n} \\ \int \frac{1}{1+x^6}\,dx &= \sum_{n=0}^{\infty} \int (-1)^n x^{6n}\,dx = \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+1}}{6n+1} + C.\end{aligned}$$

This again converges for $|x| < 1$.

Example 5.24. What is $\int_3^4 \frac{1}{1-(x-4)^3}\,dx$?

Again we have

$$\begin{aligned}
\frac{1}{1-(x-4)^3} &= \sum_{n=0}^{\infty}(x-4)^{3n} \\
\int_3^4 \frac{1}{1-(x-4)^3}\,dx &= \sum_{n=0}^{\infty}\int (x-4)^{2n}\,dx\Big|_3^4 \\
&= \sum_{n=0}^{\infty}\frac{(x-4)^{3n+1}}{3n+1}\Big|_3^4 \\
&= 0-\sum_{n=0}^{\infty}\frac{(-1)^{3n+1}}{3n+1}
\end{aligned}$$

which converges by the Alternating Series Test.

5.3 Taylor Series

In the previous section we found power series for a number of familiar functions by starting with the power series for $\frac{1}{1-x}$, and then using clever algebraic or calculus manipulations to obtain forms for our new functions. But we'd like a more systematic way of approaching the problem.

Example 5.25. One particular function we'd like a power series representation for is e^x. Let's be optimistic and assume it has one, that is let's assume

$$e^x = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + \dots$$

for some collection of constants c_i.

Note (as we saw last time) that we can evaluate the power series at 0 easily; thus $c_0 = e^0 = 1$. But how can we determine the other constants?

Well, let's take the derivative of both sides. We get

$$e^x = \sum_{n=0}^{\infty} n c_n x^{n-1} = c_1 + 2c_2 x + 3c_3 x^2 + \dots$$

and plugging in 0 for both sides gives $c_1 = e^0 = 1$. We can repeat the process; taking another derivative gives

$$e^x = 2c_2 + 6c_3 x + 12 c_4 x^2 + \dots$$

$$e^x = 6c_3 + 24 c_4 x + 60 c_5 x^2 + \dots$$

and thus $2c_2 = 1, 6c_3 = 1$, and thus $c_2 = \frac{1}{2}$ and $c_3 = \frac{1}{6}$ and more generally we have $c_n = \frac{1}{n!}$. Thus *if* we can represent e^x as a power series centered at 0, the power series must be

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

We can generalize this to any function:

Theorem 5.26. *If f has a power series representation centered at a, that is, if*

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

for some sequence of constants c_n, then $c_n = \frac{f^{(n)}(x)}{n!}$ for each n.

Definition 5.27. We define the *Taylor series* of f centered at a to be

$$T_f(x,a) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \dots$$

We sometimes say a Taylor series centered at 0 is a *Maclaurin series*, which we write

$$T_f(x,0) = T_f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 + \dots$$

Remark 5.28. It is *not true* that every function *can* be represented as a power series. If a function is not infinitely differentiable ("smooth") then it clearly doesn't have a power series, since all power series are smooth. Thus $|x|$ doesn't have a power series expansion that includes 0.

Not even all smooth functions have power series; we'll do an example later on. Functions that *can* be represented by power series are called "analytic."

But if a function can be represented by a power series, that power series is the Taylor series. Our next goal is to figure out when a function is equal to its Taylor series.

Definition 5.29. We call the truncated Taylor series the nth *Taylor polynomial* of f centered at a.

$$T_{F,n}(x,a) = T_n(x,a) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i.$$

Remark 5.30. You might notice that $T_0(x,a) = f(a)$, and $T_1(x,a) = f(a) + f'(a)(x-a)$, which might look familiar from calculus 1 as the linear approximation to f near a. The Taylor polynomials in general are an expansion of this concept; T_1 is the best linear approximation we can make to f, and T_2 is the best quadratic appoximation we can make.

In computation and in modelling we often replace a function by its Taylor polynomial to make our lives easier. We'll use this for some applications later on.

We can reframe our question—of when the Taylor series is equal to the original function—by asking when the Taylor polynomials converge to the function.

Definition 5.31. We define $R_n(x,a) = f(x) - T_n(x,a)$ to be the nth *remainder* of the Taylor series.

Then $f = T_f(x,a)$ on some interval (b,c) if and only if $\lim_{n\to+\infty} R_n(x,a) = 0$ for any x in (b,c).

Fortunately there's a way to check this, related to the Mean Value Theorem:

Proposition 5.32. *If f has enough derivatives on an interval I containing a, then for any x in I, there is a number z between x and a such that*

$$R_n(x,a) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}.$$

Note that if we take $n = 0, x = b, z = c$, we get the Mean Value Theorem.

Example 5.33. We'd like to show that the Taylor series for e^x we computed earlier actually gives us e^x. We have $f^{(n+1)}(z) = e^z$, so $R_n(x,0) = \frac{e^z}{(n+1)!}x^{n+1}$. Note that z depends on n, and z is between 0 and x, so assuming x is positive, we have

$$R_n(x,0) = \frac{e^z}{(n+1)!}x^{n+1} \leq \frac{x^{n+1}}{(n+1)!}e^x.$$

But as n goes to infinity, e^x doesn't change, and $\frac{x^{n+1}}{(n+1)!} \to 0$. So $\lim_{n\to+\infty} R_n(x,0) = 0$, and e^x is equal to its Taylor series.

Thus

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

and in particular

$$e = \sum_{n=0}^{\infty} \frac{1}{n!} = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \dots.$$

Example 5.34. We can also ask for the Taylor series of e^x centered at another number, say $a = 1$. Each derivative is still e^x and thus e^3, and so we have

$$T(x,1) = \sum_{n=0}^{\infty} e^1 \frac{(x-1)^n}{n!}.$$

Is this actually equal to e^x? We compute

$$R_n(x,1) = \frac{e^z}{(n+1)!}(x-1)^{n+1}$$

which, for any fixed x and $|z| \leq x$ goes to 0 as n goes to infinity. So we have

$$e^x = \sum_{n=0}^{\infty} e^1 \frac{(x-1)^n}{n!}.$$

This is superficially different from the previous power series, but clearly the two series give the same function. The series centered at zero will be more efficient for computing with inputs near zero, and the series centered at 1 will be more efficient for inputs near 1.

This also tells us another nice fact about e^x. Notice that if I plug 1 into this power series, I get e times what I would have gotten by plugging zero into the other power series. In general I can compute that

$$T_{x,a} = \sum_{n=0}^{\infty} e^a \frac{(x-a)^n}{n!} = e^a \sum_{n=0}^{\infty} \frac{(x-a)^n}{n!}$$

which tells me that $e^x = e^a \cdot e^{x-a}$, which is the basic arithmetic rule for multiplying exponentials.

Example 5.35. You can check that the power series we computed for $\ln(1+x)$ is the Taylor series for ln centered at 1. In particular, $(\ln x)^{(n)} = (-1)^{n-1}(n-1)!x^{-n}$.

Similarly, you can check that the power series we computed for $\arctan(x)$ is the Taylor series centered at 0.

5.4 Computing Taylor Series

First as a warmup some simple examples.

Example 5.36. We can compute the Taylor series of a polynomial. If we take the series centered at 0, we get back exactly what we started with.

Let $f(x) = x^3 + 3x^2 + 1$. Then we have $f'(x) = 3x^2 + 6x$, $f''(x) = 6x + 6$, $f'''(x) = 6$, and $f^{(n)}(x) = 0$ for $n > 3$. Thus the Taylor series centered at 0 is

$$T_f(x, 0) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{6}x^3 = 1 + 0x + \frac{6}{2}x^2 + \frac{6}{6}x^3 = 1 + 3x^2 + x^3.$$

Hopefully this is what you expected.

Probably more useful is the ability to write the Taylor series centered at a *different* point. If we take the Taylor series centered at 2, for instance, we have

$$\begin{aligned} T_f(x, 2) &= f(2) + f'(2)x + \frac{f''(2)}{2}x^2 + \frac{f'''(2)}{6} \\ &= 21 + 24(x-2) + \frac{18}{2}(x-2)^2 + \frac{6}{6}(x-2)^3 \\ &= 21 + 24(x-2) + 9(x-2)^2 + (x-2)^3. \end{aligned}$$

If you multiply this out you will get your original polynomial back; but sometimes it is very useful to have a polynomial expressed in terms of $x-2$, say, instead of in terms of x.

Example 5.37. Let's consider the function $\log x$. (We've computed a Taylor series for $\log(1+x)$ but that's a bit awkward).

If $f(x) = \log x$ then we have

$$\begin{aligned} f'(x) &= \frac{1}{x} & f''(x) &= \frac{-1}{x^2} \\ f'''(x) &= \frac{2}{x^3} & f^{(4)}(x) &= \frac{-6}{x^4} \\ &\dots \\ f^{(n)}(x) &= \frac{(-1)^{n-1}(n-1)!}{x^n} \end{aligned}$$

Thus if we wish to compute the Taylor series centered at 1, we have

$$\begin{aligned} \log(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(1)}{n!}(x-1)^n \\ &= 0 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!1^n}(x-1)^n \\ &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}(x-1)^n \end{aligned}$$

This should look familiar; it's exactly the same thing as our previous series $\log(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} x^n$, replacing x with $x-1$.

But wait, there's more! If we want to compute $\log(5)$, for instance, that power series doesn't work. But we can pick a new center for the power series and compute things there, and these power series will have different discs of convergence.

$$\begin{aligned} \log(x) &= \sum_{n=0}^{\infty} \frac{f^{(n)}(5)}{n!}(x-5)^n \\ &= \log(5) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{n!5^n}(x-5)^n \\ &= \log(5) + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n5^n}(x-5)^n. \end{aligned}$$

An application of the ratio test will show that this has radius of convergence 5, and we can see it converges conditionally on the boundary; in particular it will converge at 10 (by the alternating series test) and diverge at 0 (where it will just be the harmonic series). This behavior is in fact what we should expect: we know the series won't converge at 0 since $\log(0)$ is undefined, but we expect the series to converge everywhere it "can".

5.4.1 Trigonometry and Exponentials

While the most mysterious function we've been dealing with is e^x, we also would like to be able to compute $\sin x$ and $\cos x$.

First we'll compute the Maclaurin series for $\sin x$. Notice that $(\sin x)^{(2n)} = (-1)^n \sin x$, and $(\sin x)^{(2n+1)} = (-1)^n \cos x$. Since $\sin(0) = 0$ and $\cos(0) = 1$, the Maclaurin series is

$$T(x, 0) = 0 + x - 0 - \frac{x^3}{3!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}.$$

Computing the remainder gives

$$\begin{aligned} R_{2n}(x, 0) &= \frac{(-1)^n \cos z}{(2n+1)!} x^{2n+1} \\ |R_{2n}(x, 0)| &= \left| \frac{(-1)^n \cos z}{(2n+1)!} x^{2n+1} \right| \leq \frac{x^{2n+1}}{(2n+1)!} \end{aligned}$$

since $|\cos z| \leq 1$, and this tends to zero as n tends to infinity. So $\sin x$ is equal to its Maclaurin series, and we have

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

We also want a Maclaurin series for $\cos x$. We could compute it as we did before, but there's an easier way; $\cos x = (\sin x)'$, so

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{(2n+1)x^{2n}}{(2n+1)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!} = 1 - \frac{x^2}{2} + \frac{x^4}{4} - \frac{x^6}{6} + \dots$$

Though this is less important, sometimes we want to know things like the Taylor series for $x \sin x$. Again we'd rather not differentiate, and observe

$$\begin{aligned} \sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \\ x \sin x &= \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+2}}{(2n+1)!} = x^2 - \frac{x^4}{3!} + \frac{x^6}{5!} - \frac{x^8}{7!} + \dots \end{aligned}$$

5.4.2 The Binomial Series

Another important and widely applicable example is the *binomial series*, which is the Maclaurin series expansion for $f(x) = (1+x)^\alpha$. We can calculuate that

$$\begin{aligned}
f(x) &= (1+x)^\alpha & f(0) &= 1\\
f'(x) &= \alpha(1+x)^{\alpha-1} & f'(0) &= \alpha\\
f'''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2} & f''(0) &= \alpha(\alpha-1)\\
f'''(x) &= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3} & f'''(0) &= \alpha(\alpha-1)(\alpha-2)\\
&\vdots & &\vdots\\
f^{(n)}(x) &= \alpha(\alpha-1)\dots(\alpha-n+1)(1+x)^{\alpha-n} & f^{(n)}(0) &= \alpha(\alpha-1)\dots(\alpha-n+1)\\
&= \frac{\alpha!}{(\alpha-n)!}(1+x)^{\alpha-n} & &= \frac{\alpha!}{(\alpha-n)!}
\end{aligned}$$

So we get the formula

$$(1+x)^\alpha = \sum_{n=0}^\infty \frac{\alpha!}{(\alpha-n)!n!}x^n.$$

We sometimes use the notation

$$\binom{\alpha}{n} = \frac{\alpha!}{(\alpha-n)!n!}$$

which we read "α choose n"; if α is a positive integer this represents the number of ways to choose n things out of α choices. Then we can write

$$(1+x)^\alpha = \sum_{n=0}^\infty \binom{\alpha}{n} x^n.$$

By the ratio test, this converges when $|x| < 1$.

This series is called the binomial series, and is used very, very often to do various types of physics applications especially. Notice that if α is a positive integer this is just the usual polynomial expansion, because if α is an integer then $\binom{\alpha}{\alpha+1} = 0$.

Example 5.38. What is $(3+x)^3$? This is $3^3(1+x/3)^3$, and the binomial series gives us

$$(3+x)^3 = 27\sum_{n=0}^\infty \binom{3}{n}\left(\frac{x}{3}\right)^n = 27\left(1 + 3\cdot\frac{x}{3} + 3\cdot\frac{x^2}{9} + \frac{x^3}{27}\right) = 27 + 27x + 9x^2 + x^3.$$

Example 5.39. What is $\sqrt[3]{1+x^2}$? This is the binomial series with $\alpha = 1/3$. So the Binomial Series tells us:

$$\begin{aligned}
\sqrt[3]{1+x^2} &= \sum_{n=0}^\infty \binom{1/3}{n} x^{2n} = 1 + \frac{1}{3}x^2 + \frac{(1/3)(-2/3)}{2!}x^4 + \frac{(1/3)(-2/3)(-5/3)}{3!}x^6 + \dots\\
&= 1 + \frac{x^2}{3} - \frac{x^4}{9} + \frac{5x^6}{81} - \dots
\end{aligned}$$

Thus we can estimate, for small x, that $\sqrt[3]{1+x^2} \approx 1 + \frac{x^2}{3}$.

5.5 Applications of Taylor Series

Now that we have power series representations of a bunch of functions, we can use them to calculate limits and integrals and other messy calculus things, and in general we can use them to do lots of cool things.

5.5.1 Calculating constants

This is quick, but important. We all know that $\pi \approx 3.14$ and $e \approx 2.7$, but where do these numbers come from?

I've mentioned before that

$$e = e^1 = \sum_{n=0}^{\infty} \frac{1}{n!}$$

Summing the first four terms gives 2.708; summing the first eight gives 2.71828.

Slightly trickier is finding π. The simplest way we have to do this is observing that $\arctan(1) = \pi/4$, and then computing

$$\pi = 4\arctan(1) = 4\sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1}.$$

This series converges *very slowly* but after a few hundred terms we see 3.14 show up.

There are much better series for calculating numerical approximations of π. But this one was good enough for a particularly stubborn gentleman named Abraham Sharp to compute π to 71 digits in 1699. By hand.

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5.5.2 Integrals and Limits

Taylor series can make computing limits very easy. Heuristically when we calculate a limit we tend to ask "how many times" the top and bottom go to zero. Working with Taylor series makes this precise.

Example 5.40. What is $\lim_{x\to 0} \frac{e^x - 1 - x - x^2/2}{x^3}$?

We know that $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots$, so this is

$$\lim_{x\to 0} \frac{\frac{x^3}{3!} + \frac{x^4}{4!} + \dots}{x^3} = \lim_{x\to 0} \frac{1}{3!} + \frac{x}{4!} + \frac{x^2}{5!} + \dots = \frac{1}{6}.$$

Example 5.41. What is $\lim_{x\to 0} \frac{\sin x}{x}$?

With the same trick, we have $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$, and thus $\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \dots$. So $\lim_{x\to 0} \frac{\sin x}{x} = 1$.

Similarly, we have

$$\lim_{x\to 0} \frac{\cos x - 1}{x^2} = \lim_{x\to 0} \frac{(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots) - 1}{x^2} = \lim_{x\to 0} \frac{1}{2!} + \frac{x^2}{4!} - \frac{x^4}{6!} + \dots = \frac{1}{2}.$$

Remark 5.42. In some sense, This what L'Hôpital's Rule is "really" doing. When the top and bottom are both zero, then we take the derivative to shift both power series over one place and then try comparing again.

I've mentioned this before, but we can also use Taylor series to make difficult integrals easy.

Example 5.43. What is the integral of $x^6 \cos x$?

We can do this with integration by parts, but it's tedious. Instead, we calculate:

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{2n!}$$
$$x^6 \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+6}}{2n!}$$
$$\int x^6 \cos x \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+7}}{(2n+7)(2n!)} + C.$$

Example 5.44. There are some integrals that simply cannot be computed by normal means, such as $\int e^{-x^2} \, dx$?

It's provable that there's *no* way to represent this integral with "elementary" functions. But the integral is very important; any time you're dealing with, for instance, a normal distribution, the integral of e^{-x^2} is lurking in the background.

With our new techniques this is easy to handle:

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$$
$$\int e^{-x^2} \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(n!)} + C.$$

Thus we can compute, for instance, that

$$\int_0^1 e^{-x^2} \, dx = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)(n!)} \Big|_0^1$$
$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)(n!)} \approx .75.$$

There are still interesting questions in actually computing things with this.

Remark 5.45. A technique of complex analysis called "contour integration" tells us that $\int_{-\infty}^{+\infty} e^{-x^2}\,dx = \sqrt{\pi}$. (I told you π shows up everywhere for no reason). From this it's not too hard to show that $\int_{-\infty}^{+\infty} e^{-x^2/2}\,dx = \sqrt{2\pi}$.

Remark 5.46. This function is extremely important in probability. If you've heard of a "normal distribution" or a "bell curve" then it is represented by this function. In particular, if we choose a random number from a probability distribution with mean 0 and standard deviation 1, the probability of getting a number between a and b is $\int_a^b e^{-x^2/2}\,dx/\sqrt{2\pi}$. (The $\sqrt{2\pi}$ is to make the probability of getting "anything" equal to 1).

This makes it inconvenient that this integral is difficult, but extremely useful that we can now approximate it numerically.

5.5.3 Approximating functions

The primary use of Taylor series is to conduct approximate calculations of functions we can't or don't want to calculate exactly.

Example 5.47. What is $\sqrt[n]{1+x}$ when x is small?

When we look at this we should immediately think of the binomial series (with $\alpha = 1/n$). Thus

$$\begin{aligned}\sqrt[n]{1+x} &= \sum_{k=0}^{\infty} \binom{1/n}{k} x^k = 1 + \frac{1}{n}x + \frac{(1/n)((1-n)/n)}{2}x^2 + \dots \\ &\approx 1 + \frac{x}{n}.\end{aligned}$$

Note that this approximation works better when x is small. But we can, as a rule of thumb, approximate $\sqrt[n]{2} \approx \frac{n+1}{n}$.

Example 5.48. Approximate $\sqrt[5]{x}$ near 32 to degree two.

We have two options. The first is to turn this into the binomial series

$$\begin{aligned}\sqrt[5]{32+x} &= 2(1+\tfrac{x}{32})^{1/5} = 2\sum_{n=0}^{\infty}\binom{1/5}{n}x^n \\ &= 2 + \frac{2}{5}(x/32) - \frac{4}{25}(x/32)^2 + \dots\end{aligned}$$

(Note this converges when $|x/32|1$ and thus when $-32 < x < 32$). So we can approximate

$$\sqrt[5]{32+x} \approx 2 + \frac{x}{80} - \frac{x^2}{6400}.$$

Alternately, we could compute the Taylor polynomial anew, centered at 32:

$$\begin{aligned} f(x) &= \sqrt[5]{x} & f(32) &= 2 \\ f'(x) &= \frac{1}{5}x^{-4/5} & f'(32) &= \frac{1}{80} \\ f''(x) &= \frac{-4}{25}x^{-9/5} & f''(32) &= \frac{-1}{3200} \end{aligned}$$

and thus

$$\begin{aligned} \sqrt{x} &= \sum_{n=0}^{\infty} \frac{f^{(n)}(32)}{n!}x^n \\ &\approx 2 + \frac{x}{80} - \frac{x^2}{3200 \cdot 2}. \end{aligned}$$

Either way, we can estimate $\sqrt[5]{36} \approx 2 + \frac{4}{80} - \frac{16}{6400} = 2 + \frac{1}{16} - \frac{1}{800} \approx 2.06$.

Example 5.49 (Pendulums and Clocks)**.** One place we often use Taylor approximations is in modelling physical systems, such as a pendulum.

We use pendulums in clocks (e.g. grandfather clocks) becaues they keep accurate time. The principle underlying this is the idea that a given pendulum takes the same amount of time to complete a swing regardless of the size of that swing.

This is, unfortunately, false. The angular acceleration on a pendulum (that is, how quickly it changes the angle of rotation) is given by $\alpha = -\frac{mg}{L}\sin\theta$, and working out the length of time it takes the pendulum to complete a swing is a nasty integral that doesn't have a closed-form answer, known as an "elliptic integral." (It's called this because it's also the type of integral used to calculate the circumference of an ellipse).

But you may notice that most clocks have a long pendulum that only makes small arcs. When the angle of the pendulum is small, we can use the Taylor series of sin to approximate $\sin\theta \approx \theta$, and then we have $\alpha \approx -\frac{mg}{L}\theta$. Since we've removed the $\sin\theta$ we can now compute the integral in question, and we get that the time a pendulum takes to complete one swing is $T \approx 2\pi\sqrt{L/g}$. (The error in this approximation causes a typical grandfather clock to lose about 15 seconds a day).

(Incidentally, if we need to know the exact answer, we can get it by, again, integrating the Taylor series. We get

$$T = 2\pi\sqrt{L/g}\left(1 + \frac{\theta^2}{16} + \frac{11}{3072}\theta^4 + \dots\right).$$

Example 5.50 (Springs in physics)**.** More generally,as you spend more time doing physics, you'll discover that almost every system you want to study is modeled as a simple spring.

A spring is a system governed by a simple quadratic equation. So any system governed by a quadratic equation can be treated as a spring. So if you take any system and approximate it with the second Taylor polynomial, you get something that looks like a spring.

As examples: we often model light interacting with matter by treating the atom as an electron on a spring. The strength of the chemical bond between two atoms is given by the Lennard-Jones Potential, which is $\left(\frac{R_m}{r}\right)^{12} - 2\left(\frac{R_m}{r}\right)^6$. If we take the Taylor series centered at R_m we get $-1 + 0r + \frac{72}{R_m^2}r^2 + \cdots \approx \frac{72}{R_m^2}r^2 - 1$, which is just a parabola.

As a note: we often refer to this process as "dropping higher order terms"—the constant term is the order-0 term; the linear term is the order-1 term; we have dropped every term of order higher than 2. We sometimes abbreviate even further and call them the H.O.T.

Example 5.51 (Relativity)**.** The last example we want to do concerns special relativity. Relativity concerns a number of interesting phenomena that occur when one's velocity is relatively large compared to the speed of light. But we know that at low velocities, special relativity should "look like" Newtonian mechanics.

Most of the relativity equations feature a variable γ, where $\gamma = \frac{1}{\sqrt{1-(v/c)^2}}$. We'd like to use the binomial expansion, so we write

$$\gamma = (1 - (v/c)^2)^{-1/2} \approx 1 + \frac{-1}{2}(-(v/c)^2) = 1 + \frac{1}{2}\frac{v^2}{c^2}$$

is the first-order Taylor approximation to γ. It should be accurate when v/c is small–that is, when our velocity is very small relative to the speed of light.

For instance, relativity says the kinetic energy of an object is $E = mc^2(\gamma - 1)$. It's clear that when $v = 0$ then this is 0. If v is small, we can take the Taylor expansion from before, and get

$$E \approx mc^2\left((1 + \frac{1}{2}\frac{v^2}{c^2} - 1\right) = \frac{1}{2}mv^2$$

which is the usual Newtonian formula for kinetic energy.

Similarly, the formula for time dilation is $T' = T\gamma$. If we take the first-order approximation, we have $T' = T + \frac{T}{2}\frac{v^2}{c^2}$. But even better, if we take the zeroth-order approximation, we have $\gamma \approx 1$ and thus $T' \approx T$. This tells us that at low velocities, time dilation is negligible.

5.6 Bonus Taylor Series Fun

5.6.1 Power Series and Differential Equations

In section 3.4 we talked about solving separable differential equations, but most differential equations are not separable. There are a lot of tools we can use to solve non-separable equations, but one approach is to use power series—which basically always works.

Example 5.52. Recall the classic differential equation $y' = y$. Suppose the function $y(x)$ can be represented by a power series. Then we have

$$y = \sum_{n=0}^{\infty} a_i x^i = a_0 + a_1 x + a_2 x^2 + \dots$$
$$y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots$$

and since these are the same function, each coefficient has to match up. So we get the system

$$\begin{aligned} a_1 &= a_0 & \qquad a_1 &= a_0 \\ 2a_2 &= a_1 & a_2 &= a_1/2 \\ 3a_3 &= a_2 & a_3 &= a_2/3 \\ &\vdots & &\vdots \end{aligned}$$

So if we know a_0, then we can figure out all the other coefficients.

How do we find a_0? Well, that's a choice. Remember any differential equation will wind up with free constants in the end. But if we take $a_0 = 1$, which seems like a reasonable choice, we then get

$$\begin{aligned} a_1 &= a_0 = 1 \\ a_2 &= a_1/2 = 1/2 \\ a_3 &= a_2/3 = 1/6 \\ &\vdots \\ a_n &= a_{n-1}/n = 1/n! \\ y &= 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \dots \end{aligned}$$

which is exactly the Taylor series for e^x we worked out in section 5.3.

(What happens if we choose a different a_0? That just multiplies all the coefficients by a constant, so if $a_0 = C$ then our power series gives us Ce^x. And we already know that Ce^x is the solution to this differential equation!)

Example 5.53. Solve $y'' - 3xy' + y = 0$.

We don't have any tools for this, so we use Taylor series. Assume $y = \sum_{n=0}^{\infty} c_n x^n$, and then we have

$$\begin{aligned}
\left(\sum_{n=0}^{\infty} c_n x^n\right)'' - 3x\left(\sum_{n=0}^{\infty} c_n x^n\right)' + \sum_{n=0}^{\infty} c_n x^n &= 0 \\
\sum_{n=0}^{\infty} c_n n(n-1)x^{n-2} - 3x\sum_{n=0}^{\infty} c_n n x^{n-1} + \sum_{n=0}^{\infty} c_n x^n &= 0 \\
\sum_{n=0}^{\infty} c_{n+2}(n+1)(n+2)x^n - 3\sum_{n=0}^{\infty} c_n n x^n + \sum_{n=0}^{\infty} c_n x^n &= 0 \\
\sum_{n=0}^{\infty} \left((n+1)(n+2)c_{n+2} - 3nc_n + c_n\right) x^n &= 0
\end{aligned}$$

and thus for each n we have

$$\begin{aligned}
(n+1)(n+2)c_{n+2} &= (3n-1)c_n \\
c_{n+2} &= \frac{(3n-1)c_n}{(n+1)(n+2)}
\end{aligned}$$

as our recurrence relation. As before, we see that our solution must have the form

$$\begin{aligned}
y &= \sum_{k=0}^{\infty} \frac{c_0((5)(11)\cdots(6k-1))}{(2k)!}x^{2k} + \frac{c_1((8)(14)\cdots(6k+2)}{(2k+1)!}x^{2k+1} \\
&= c_0 \sum_{k=0}^{\infty} \frac{(5)(11)\cdots(6k-1)}{(2k)!}x^{2k} + c_1 \sum_{k=0}^{\infty} \frac{(8)(14)\cdots(6k+2)}{(2k+1)!}x^{2k+1}.
\end{aligned}$$

Example 5.54. In section 5.1 I mentioned the *Bessel function*

$$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}.$$

This arises naturally as a solution to the differential equation

$$x^2y'' + xy' + x^2y = 0$$

which is used to study a lot of physics on cylinders.

5.6.2 Taylor Series and Complex Numbers

Perhaps the most surprising and important fact about the trigonometric power series is the way they combine. You'll notice that for both sin and cos, every term looks like $\frac{x^n}{n!}$ but neither series has all the terms.

Leonhard Euler, around 1740, asked himself what it means to exponentiate an imaginary number. Since the Taylor series of e^x agrees with the function everywhere on the real line, it makes sense to define $e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. If we plug in a purely imaginary number ix for $x \in \mathbb{R}$, we see:

$$\begin{aligned} e^{ix} &= \sum_{n=0}^{\infty} \frac{(ix)^n}{n!} = \sum_{n=0}^{\infty} \frac{i^{2n} x^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{i(i)^{2n} x^{2n+1}}{(2n+1)!} \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \cos(x) + i\sin(x). \end{aligned}$$

Thus we obtain *Euler's Formula*:

$$e^{i\theta} = cos\theta + i\sin\theta.$$

As a corollary, we get the statement Euler called the most beautiful in all mathematics: $e^{i\pi} = \cos(\pi) + i\sin(\pi) = 1$, or

$$e^{i\pi} - 1 = 0.$$

This statement relates the five most fundamental constants in the complex plane.

But in addition to being really pretty, this also has a geometric interpretation. If we think about the point $\cos\theta + i\sin\theta$ on the complex plane, it has x-coordinate $\cos\theta$ and y-coordinate $\sin\theta$ and is thus the point on the unit circle corresponding to angle θ. (This is why the unit circle is oriented as it is!)

So in general we can represent any point on the unit circle as $e^{i\theta}$, where θ is the angle the point makes from the positive x-axis. Further, if we have *any* complex number z, we can represent it in "polar coordinates" by giving its absolute value $|z|$ and its complex argument θ. Thus for any complex number z, we have

$$z = |z|e^{i\arg(z)}.$$

Another result from this line of thinking is De Moivre's Formula:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

There are some functions, unfortunately, that don't behave nearly so nicely, such as the logarithm. Unlike exp and sin and cos we can't just extend the power series for log to the whole complex plane, since it has finite radius of convergence (and an unavoidable pole at 0).

But in fact the problem is deeper. We generally want to define log to be the inverse of exp, so that $\log\exp z = z$ for every z. But notice that $\exp(0) = 1$, and also $\exp(2\pi i) = 1$. So if we want to define log on the whole complex plane we must have $\log 1 = 0$ and also

$\log 1 = 2\pi i$ and this is obviously a problem, since functions can't have multiple outputs. We in fact have infinitely many numbers z with $e^z = 1$, or in fact any complex number you choose (except 0; $\log 0$ is never defined).

We solve this by choosing a "branch," which basically corresponds to which complex arguments we allow; we will typically require our arguments to be in $(-\pi, \pi]$; this is called the "principal value" of the argument. (Notice this is similar to the way we require $\arcsin x$ to be in $[-\pi, \pi]$ in section 1.5). Notice also that in this case, we aren't really happy at negative real numbers–there's a huge jump discontinuity there in the complex argument.

There is another option, which avoids this discontinuity, and which I mention mainly because it's cool. We can define what's called a *Riemann Surface*, which is a two-dimensional surface that we can think of as sitting in three-dimensional space. In this case our Riemann surface looks like a giant helical Archimedes Screw.

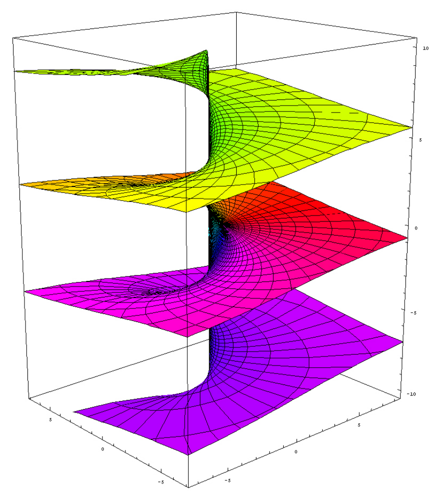

Figure 5.1: The Riemann surface corresponding to the complex logarithm

This surface "covers" the complex plane with infinitely many "sheets." The logarithm is a function defined on this surface; which sheet we are in tells us which "branch" the argument should be in, and other than that the logarithm is defined as it would be for the point of the complex plane "under" our surface. Thus the logarithm of a point on one sheet would be 0, and the logarithm of the point one sheet above it is $2\pi i$, and if we go up another sheet we get $4\pi i$, and so on.

This behavior also occurs with functions like $\sqrt[n]{z}$. Since every (non-zero) number has two square roots, the square root function is "doubly ramified" or a "two-fold cover" of the complex plane. The nth-root function is an n-fold cover. They are, in fact, essentially the same picture as the logarithm picture, but with only finitely many sheets, which wrap around and join up. By convention, we put the discontinuity still on the negative real line.

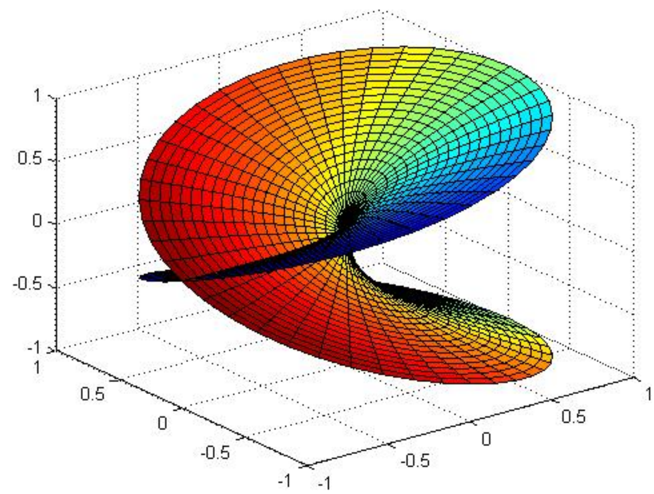

Figure 5.2: The Riemann surface corresponding to the square root function

Special attention should be paid to the idea of an "nth root of unity", where "unity" is just a fancy word for "one." We know that one should have n nth roots; a little thought shows that they are $e^{2\pi i/n}$, since $\left(e^{2\pi i/n}\right)^n = e^{2\pi i} = 1$. These are points spaced evenly around the unit circle. They are very useful in creating functions and other operations that have certain types of "periodicity", which means they repeat every n times.

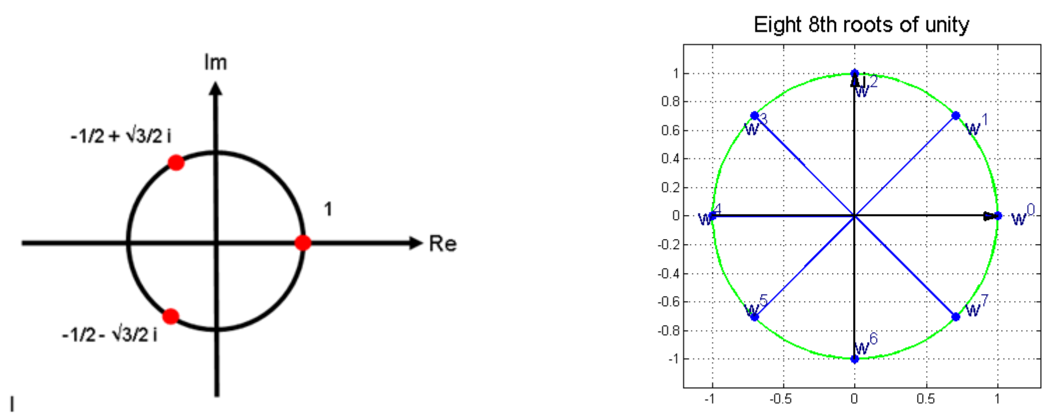

Figure 5.3: Cube Roots and Eighth Roots of Unity

5.6.3 Failure Modes of Taylor Series

Sadly, while Taylor series are awesome, they don't always work. Consider the function defined by $f(x) = e^{-1/x^2}$ and $f(0) = 0$. This function is continuous and in fact differentiable at 0:

$$\lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{e^{-1/h^2}}{h} = 0.$$

We can repeat this work, and we see that $f''(0) = f'''(0) = \cdots = f^{(n)}(0) = 0$. Thus the Taylor series is

$$T_f(x, 0) = \sum_{n=0}^{\infty} \frac{0}{n!} x^n = 0.$$

But clearly $f(x) \neq 0$ when $x \neq 0$, so f is equal to its Taylor series only in the trivial case when $x = 0$. So just remember: Taylor series *don't always work*.

5.7 Double Bonus: Real Fourier Series

We've seen that we can represent many functions as a power series, an infinite sum of multiples of powers of x. But some functions are hard to represent this way, and we want other tools. In particular we can represent a function as an infinite sum of trigonometric functions.

For this discussion we'll confine ourselves to real functions on the interval $[-\pi, \pi]$. (The same idea works for functions on any closed interval, but it's easier to talk about just this particular interval for right now. Also, π has popped up again for no reason. Hi, π!)

Definition 5.55. Let f be a function on the interval $[-\pi, \pi)$. Then the *Fourier series* of f is given by

$$f_\infty(x) = \frac{C}{2} + \sum_{n=1}^{\infty} (a_n \sin(nx) + b_n \cos(nx))$$

where

$$C = 2\langle f(x), 1 \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x)\, dx$$
$$a_n = 2\langle f(x), \sin(nx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin(nx)\, dx$$
$$b_n = 2\langle f(x), \cos(nx) \rangle = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos(nx)\, dx.$$

For Taylor series we could match things up by taking derivatives; to find Fourier series coefficients we instead compute integrals. That makes the computations much nastier! But the computations are *doable*, and they make sense because the integral of one term times a different term is always zero.

Before we dive into computations to prove this, let's think about why we should expect it to be true. A sin or cos function passes through a complete cycle between $-\pi$ an π, so the positive bits will exactly cancel out the negative bits. When we multiply two different sin or cos functions, they don't correlate with each other—each one passes through cycles at a different rate from the others, so the cycles don't reinforce or cancel out. Thus we'll still have exactly as much on top as we do on bottom, and the integrals should be zero.

Proof. We use the notation $\langle f, g\rangle$ to represent $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx$.

$$2\pi\langle 1, \sin(nx)\rangle = \int_{-\pi}^{\pi} \sin(nx)\,dx = \left.\frac{-\cos(nx)}{n}\right|_{-\pi}^{\pi} = \frac{1}{n} - \frac{1}{n} = 0.$$
$$\langle 1, \cos(nx)\rangle = \int_{-\pi}^{\pi} \cos(nx)\,dx = \left.\frac{\sin(nx)}{n}\right|_{-\pi}^{\pi} = 0 - 0 = 0.$$

The products of the sin and cos functions are a bit trickier.

$$\begin{aligned}\langle \sin(nx), \sin(mx)\rangle &= \int_{-\pi}^{\pi} \sin(nx)\sin(mx)\,dx \\ &= -\frac{\cos(nx)}{n}\sin(mx)\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{m}{n}\cos(nx)\cos(mx)\,dx \\ &= 0 + \frac{m}{n^2}\sin(nx)\cos(mx)\Big|_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{m^2}{n^2}\sin(nx)\sin(mx)\,dx \\ 0 &= \left(\frac{m^2}{n^2} - 1\right)\int_{-\pi}^{\pi} \sin(nx)\sin(mx)\end{aligned}$$

As long as $\frac{m^2}{n^2} \neq 1$ this implies that $\langle \sin(nx)\sin(mx)\rangle = 0$. For positive integers m, n, this holds whenever $m \neq n$. In contrast

$$\begin{aligned}\int_{-\pi}^{\pi} \sin^2(nx)\,dx &= \int_{-\pi}^{\pi}\frac{1-\cos(2x)}{2}\,dx \\ &= \left(\frac{x}{2} - \frac{\sin(2nx)}{4}\right)\Big|_{-\pi}^{\pi} = \left(\frac{\pi}{2} - \frac{-\pi}{2}\right) = \pi. \\ \langle \sin(nx), \sin(nx)\rangle &= \frac{1}{2\pi}\cdot\pi = \frac{1}{2}.\end{aligned}$$

Similar arguments work for $\langle\cos(nx), \sin(mx)\rangle$ and $\langle\cos(nx), \cos(mx)\rangle$. □

Theorem 5.56. *Suppose f is a continous function with continuous derivative, except for finitely many points, on $[-\pi, \pi)$. Then $f(x)$ is equal to its Fourier series except for at finitely many points.*

Notice that unlike in the case of Taylor series, this always works. every *continuous function is (essentially) equal to its Fourier series.*

What does this mean? It means that if we have a function on $[-\pi, \pi)$ then we can look at it as being composed of a bunch of different "waves" of different frequencies, and the coefficients tell us how large each wave is. (The constant term tells us the average value around which the waves are oscillating). Further, a Fourier series is always a periodic function on the whole real line. So any periodic function can be viewed as a Fourier series, and this technology allows us to see it as composed of many smaller simpler waves. We'll return to the physics and geometry of this soon.

Example 5.57. Let $f(x) : [-\pi, \pi) \to \mathbb{R}$ be given by $f(x) = x$. The periodic version of this function is a "sawtooth wave." Then we have:

$$\begin{aligned}
\frac{C}{2} &= \langle f(x), 1 \rangle \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx \\
&= \frac{1}{2\pi} \int_{-\pi}^{\pi} x dx \\
&= \frac{1}{2\pi} x^2 |_{-\pi}^{\pi} = 0.
\end{aligned}$$

$$\begin{aligned}
a_n &= 2\langle f(x), \sin(nx) \rangle \\
&= \frac{1}{\pi} \int_{-\pi}^{\pi} x \sin(nx) dx \\
&= \frac{1}{\pi} \left(-x \cdot \frac{\cos nx}{n} |_{-\pi}^{\pi} + \int_{-\pi}^{\pi} \frac{\cos(nx)}{n} dx \right) \\
&= \frac{1}{\pi} \left(\frac{-\pi \cos(n\pi) - \pi \cos(-n\pi)}{n} \right) \\
&= -2 \frac{\cos(n\pi)}{n} = (-1)^{n+1} \frac{2}{n}.
\end{aligned}$$

$$\begin{aligned}
b_n &= 2\langle f(x), \cos(nx)\rangle \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} x\cos(nx)dx \\
&= \frac{1}{\pi}\left(x \cdot \frac{\sin nx}{n}|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} \frac{\sin(nx)}{n}dx\right) \\
&= \frac{1}{\pi}\left(\frac{\pi\sin(n\pi) + \pi\sin(-n\pi)}{n}\right) \\
&= 0.
\end{aligned}$$

Thus the sawtooth wave has Fourier series

$$f(x) = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{2}{n}\sin(nx).$$

Example 5.58. Define sgn : $[-\pi, \pi] \to \mathbb{R}$ by $f(x) = -1$ if $x < 0$ and $f(x) = 1$ if $x \geq 0$. (Made periodic, this is a "square wave").

$$\begin{aligned}
\frac{C}{2} &= \langle \operatorname{sgn}(x), 1/2\rangle \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{sgn}(x)dx = 0.
\end{aligned}$$

$$\begin{aligned}
a_n &= \frac{2}{2\pi}\langle \operatorname{sgn}(x), \sin(nx)\rangle \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} -\sin(nx)dx + \int_{0}^{\pi}\sin(nx)dx\right) \\
&= \frac{1}{\pi}\left(\frac{\cos(nx)}{n}|_{-\pi}^{0} - \frac{\cos(nx)}{n}|_{0}^{\pi}\right) \\
&= \frac{1}{n\pi}(\cos(0) - \cos(-n\pi) - \cos(n\pi) + \cos(0)) \\
&= \frac{2}{n\pi}(\cos(0) - \cos(n\pi))
\end{aligned}$$

which equals $\frac{r}{n\pi}$ if n is odd and 0 if n is even.

$$\begin{aligned}
b_n &= \frac{2}{2\pi}\langle \operatorname{sgn}(x), \cos(nx)\rangle \\
&= \frac{1}{\pi}\left(\int_{-\pi}^{0} -\cos(nx)dx + \int_0^{\pi} \cos(nx)dx\right) \\
&= \frac{1}{\pi}\left(\frac{-\sin(nx)}{n}|_{-\pi}^{0} + \frac{\sin(nx)}{n}|_0^{\pi}\right) \\
&= \frac{1}{\pi n}(-\sin(0) + \sin(-n\pi) + \sin(n\pi) - \sin(0)) = 0.
\end{aligned}$$

Thus

$$\operatorname{sgn}(x) = \sum_{n=0}^{\infty} \frac{4}{(2n+1)\pi} \sin((2n+1)x).$$