Math 1232 Sprin 2021
Single-Variable Calculus II Mastery Quiz 10
Due Friday, April 9

This week's mastery quiz has ten topics. You should do topics 20 and 19, and optionally *one* of the previous topics. Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 20-30 minutes on this quiz.

Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file.* You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

20. Power Series as Functions
19. Power Series
18. Absolute and Conditional Convergence
17. Comparison Test and Limit Comparison Test
16. Divergence and Integral tests
15. Geometric and Telescoping Series
14. Sequences
13. Separable Differential Equations
9. Partial Fractions
7. Integration by Parts

20. **Power Series as Functions**

(a) Write a power series expression for $\frac{2x^2}{4x+1}$ centered a 0. What is the radius of convergence?

Solution: We know that

$$\begin{aligned} \frac{1}{1-(-4x)} &= \sum_{n=0}^{\infty}(-4x)^n \\ \frac{2x^2}{1+4x} &= 2x^2\sum_{n=0}^{\infty}(-4)^n x^n \\ &= \sum_{n=0}^{\infty} 2\cdot(-4)^n x^{n+2} \\ \text{(or)} \quad &= \sum_{n=2}^{\infty} 2^{2n-3}(-1)^n x^n. \end{aligned}$$

The radius of convergence is $1/4$. We can figure that out by reasoning from the geometric series: the radius of convergence for the geometric series is 1, so it converges for $-1 < -4x < 1$ or $-1/4 < x < 1/4$. Or we can use the ratio test:

$$\lim_{n\to\infty}\left|\frac{2^{2n-1}(-1)^{n+1}x^{n+1}}{2^{2n-3}(-1)^n x^n}\right| = \lim_{n\to\infty} 4|x|$$

and thus it converges when $4|x| < 1$.

(b) If $f(x) = \sum_{n=0}^{\infty}\frac{n+1}{n!+1}x^n$, compute $\int_3^5 f(x)$.

Solution:

$$\begin{aligned} \int f(x) &= \sum_{n=0}^{\infty}\frac{1}{n!+1}x^{n+1} + C \\ \int_3^5 f(x) &= \sum_{n=0}^{\infty}\frac{1}{n!+1}(5^{n+1}-3^{n+1}). \end{aligned}$$

19. **Power Series**

(a) Find the radius of convergence and the interval of convergence of $\sum_{n=0}^{\infty}\frac{(2x-5)^n}{n^2}$.

Solution:

We use the ratio test.

$$\begin{aligned} \lim_{n\to\infty}\left|\frac{(2x-5)^{n+1}/(n+1)^2}{(2x-5)^n/n^2}\right| &= \lim_{n\to\infty}\left|\frac{(2x-5)n^2}{(n+1)^2}\right| \\ &= |2x-5|\lim_{n\to\infty}\frac{n^2}{(n+1)^2} = |2x-5|. \end{aligned}$$

So we need $|2x-5| < 1$ or $-1 < 2x-5 < 1$, or $4 < 2x < 6$ or $2 < x < 3$. So the radius is $1/2$.

To find the interval we need to check the endpoints. We see

$$\sum_{n=0}^{\infty} \frac{(4-5)^n}{n^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{n^2}$$

converges by alternating series test

$$\sum_{n=0}^{\infty} \frac{(6-5)^n}{n^2} = \sum_{n=0}^{\infty} \frac{1}{n^2}$$

converges by p-series test

(b) Find the radius of convergence and the interval of convergence of $\displaystyle\sum_{n=0}^{\infty} \frac{n^2 x^n}{1 \cdot 3 \cdot 5 \cdot \dots \cdot (2n-1)}$.

Solution:

We use the ratio test.

$$\lim_{n \to \infty} \left| \frac{(n+1)^2 x^{n+1} / 1 \cdot 3 \cdot \dots \cdot (2n+1)}{n^2 x^n / 1 \cdot 3 \cdot \dots \cdot (2n-1)} \right| = \lim_{n \to \infty} \left| \frac{(n+1)^2 x}{n^2 (2n+1)} \right|$$
$$= |x| \lim_{n \to \infty} \frac{(n+1)^2}{n^2(2n+1)} = 0.$$

This is always less than 1, so the series always converges. The radius of convergence is ∞ and the interval of convergence is $(-\infty, \infty)$.

18. **Absolute and Conditional Convergence**

For each series, tell whether it absolutely converges, conditionally converges, or diverges. Justify your answer (and in particular, if it conditionally converges, explain why it doesn't absolutely converge).

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{(-2)^n}{n^3+n}$

Solution:

We use the Ratio test. We have

$$\lim_{n \to \infty} \left| \frac{(-2)^{n+1} / (n+1)^3 + n + 1}{(-2)^n / n^3 + n} \right| = \lim_{n \to \infty} \frac{2(n^3+n)}{(n+1)^3 + n + 1}$$
$$= \lim_{n \to \infty} 2 > 1.$$

This limit is greater than 1, so by the ratio test this diverges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n (3n^2+5n+2)^n}{(5n^2-3)^n}$

Solution:

We use the root test. We have

$$\lim_{n \to \infty} \sqrt[n]{\left| \frac{(-1)^n (3n^2+5n+2)^n}{(5n^2-3)^n} \right|} = \lim_{n \to \infty} \frac{3n^2+5n+2}{5n^2-3} = \frac{3}{5} < 1$$

So by the root test this converges absolutely..

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n\sqrt{n}}{2n+3}$

Solution:

This is an alternating series. Since the terms $\frac{\sqrt{n}}{2m+3}$ tend to zero as n goes to infinity, this converges by the alternating series test.

However, it doesn't absolutely converge. If we look at the absolute value series, we have $\sum_{n=1}^{\infty} \frac{\sqrt{n}}{2n+3}$. You can see this doesn't converge in a couple ways. The integral test isn't super plausible here. You can do a comparison test to $\frac{1}{\sqrt{n}}$: this is larger than $\frac{1}{3\sqrt{n}}$ for large n, and $\frac{1}{3\sqrt{n}}$ diverges. (note: this is *not* larger than $\frac{1}{\sqrt{n}}$!)

It may be easier to use the limit comparison test, though. We have

$$\lim_{n\to\infty} \frac{\sqrt{n}/2n+3}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{n}{2n+3} = 1/2.$$

Since the series $\sum \frac{1}{\sqrt{n}}$ diverges, by the limit comparison test, $\sum \frac{\sqrt{n}}{2n+3}$ diverges, and thus our series does not converge absolutely.

17. **Comparison Tests**

Determine whether each of the following series converges by using an appropriate comparison test.

(a) $\sum_{n=1}^{\infty} \frac{n^3+n-1}{n^5-3n^4}$

Solution:

Here it would be hard to use the regular comparison test. It's true that $\frac{n^3+n-1}{n^5-3n^4} \geq \frac{1}{n^2}$, but since this says it's greater than a convergent series, it doesn't really help. Instead, we limit compare to $\frac{1}{n^2}$. We have

$$\lim_{n\to\infty} \frac{n^3+n-1/n^5-3n^4}{1/n^2} = \lim_{n\to\infty} \frac{n^5+n^3-n^2}{n^5-3n^4} = 1.$$

Since this is a finite non-zero number, the two series have the same convergence behavior. Thus, since $\frac{1}{n^2}$ converges, we know that our series also converges.

(b) $\sum_{n=1}^{\infty} \frac{\ln(n)+n}{n^2-1}$

Solution: You can't really use the limit comparison test here, at least not easily, because the numerator is a bit over-complicated. But you can use the usual comparison test. We know that $n \leq n+\ln(n)$ and $n^2-1 < n^2$, so

$$\frac{\ln(n)+n}{n^2-1} \geq \frac{n}{n^2} = \frac{1}{n}.$$

We know that $\sum \frac{1}{n}$ diverges by the p-series test, so our series diverges by the comparison test.

16. **Divergence and Integral Tests**

Determine whether each of the following series converges or diverges. Justify your answers using only the divergence and integral tests (and *not* the comparison tests or ratio test or root test).

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+4}$

Solution: We can work this out with the integral test. We have

$$\begin{aligned}\int_1^{\infty} \frac{1}{x^2+4}\,dx &= \lim_{t\to\infty} \int_1^t \frac{1}{4}\frac{1}{1+(x/2)^2}\,dx \\ &= \lim_{t\to\infty} \frac{1}{2}\arctan(x/2)\Big|_1^t \\ &= \lim_{t\to\infty} \arctan(t/2) - \arctan(1/2) = \pi/2 - \arctan(1/2) < \infty.\end{aligned}$$

Since this integral converges, the series must also converge by the integral test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^2+5}{5n^2-3n+2}$

Solution: I do not want to try to integrate this. But $\lim_{n\to\infty} \frac{3n^2+5}{5n^2-3n+2} = 3/5 \neq 0$, so by the divergence test this series diverges.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{n+1}{n^2+2n}$

Solution: We take $u = x^2+2x$ so that $du = 2x+2\,du$, and we have

$$\begin{aligned}\int_1^{\infty} \frac{x+1}{x^2+2x}\,dx &= \lim_{t\to\infty} \int_1^t \frac{x+1}{x^2+2x}\,dx \\ &= \lim_{t\to\infty} \frac{1}{2}\ln|x^2+2x|\Big|_1^t \qquad = \frac{1}{2}\lim_{t\to\infty} \ln|t^2+2t| - \ln(3) = \infty.\end{aligned}$$

Since this improper integral is infinite, by the integral test the series does not converge.

15. **Geometric and Telescoping Series**

Compute the following infinite sums, with justification:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{7^{n+1}}{2\cdot 3^{2n-1}} =$

Solution:

This is a geometric series with $a = \frac{49}{6}$ and $r = \frac{7}{9}$. So the sum is

$$\frac{49/6}{1-7/9} = \frac{49/6}{2/9} = \frac{147}{4}.$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-3)^n}{2^n} =$

Solution:
This is a geometric series with $a = \frac{-3}{2}$ and $r = \frac{-3}{2}$. Since $|r| = 3/2 > 1$ this series does not converge.

(c) $\displaystyle\sum_{n=1}^{\infty} \ln\left(\frac{n+4}{n+3}\right) =$

Solution: This is the same as

$$\begin{aligned}\sum_{n=1}^{\infty} \ln(n+4) - \ln(n+3) &= \lim_{t\to\infty} \sum_{n=1}^{t} \ln(n+4) - \ln(n+3) \\ &= \lim_{t\to\infty} \big(\ln(5) - \ln(4)\big) + \big(\ln(6) - \ln(5)\big) + \cdots + \big(\ln(t+4) - \ln(t+3)\big) \\ &= \lim_{t\to\infty} \ln(t+4) - \ln(4) = \infty.\end{aligned}$$

Thus this sum diverges to infinity.

14. **Sequences**

(a) Consider the sequence $(a_n) = (3, {}^6/_2, {}^9/_6, {}^{12}/_{24}, {}^{15}/_{120}, \dots)$. Find a formula for the nth term a_n. Compute $\lim_{n\to\infty} a_n$.

Solution:
We have $a_n = \frac{3n}{n!}$, and thus

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{3n}{n!} = \lim_{n\to\infty} \frac{1}{(n-1)!} = 0.$$

(b) Let $b_n = \tan\left(\frac{(2n-1)\pi}{4}\right)$. Compute the first four terms of the sequence, and compute $\lim_{n\to\infty} b_n$, with justification.

Solution:

$$b_1 = \tan(\pi/4) = 1 \qquad b_2 = \tan(3\pi/4) = -1$$
$$b_3 = \tan(5\pi/4) = 1 \qquad b_4 = \tan(7\pi/4) = -1.$$

We see that this sequence just alternates between 1 and -1, so it will never converge. The limit does not exist.

(c) Let $c_n = \frac{5n+1}{3n-2}$. Compute the first four terms of this sequence, and compute $\lim_{n\to\infty} c_n$, with justification.

Solution:

$$c_1 = 6 \qquad c_2 = {}^{11}/_4$$
$$c_3 = {}^{16}/_7 \qquad c_4 = {}^{21}/_{10}$$

We can look at this as a function and use L'Hospital's rule:

$$\lim_{x\to\infty} \frac{5n+1}{3n-2} = \lim_{x\to\infty} \frac{5}{3}.$$

Thus the limit of our sequence is ${}^5/_3$.

13. **Differential Equations**

(a) Find a general solution to the equation $y' = x^2/y^3$.
Solution:

$$\begin{aligned} \frac{dy}{dx} &= x^2/y^3 \\ y^3\,dy &= x^2\,dx \\ y^4/4 &= x^3/3 + Cy \qquad\qquad = \sqrt[4]{x^3/12 + C/4}. \end{aligned}$$

(b) Find a (specific) solution to the initial value problem $y' = xy - x$ if $y(0) = e + 1$
Solution:

$$\begin{aligned} \frac{dy}{dx} &= xy - x \\ \frac{dy}{y-1} &= x\,dx \\ \ln|y-1| &= x^2/2 + Cy \qquad\qquad = e^{x^2/2+C} + 1 \end{aligned}$$

Then we have

$$\begin{aligned} e + 1 &= e^{0^2/2+C} + 1 \\ 1x^2/2 + CC & \qquad\qquad = 1 \\ y &= e^{x^2/2+1} + 1. \end{aligned}$$

9. **Partial Fractions**

(a) Compute $\int \frac{x^2+x}{x-1}\,dx =$
Solution: Polynomial long division gives that

$$\begin{aligned} x^2 + x &= x(x-1) + 2(x-1) + 2 \\ \frac{x^2+x}{x-1} &= x + 2 + \frac{2}{x-1} \\ \int \frac{x^2}{x-1}\,dx &= \int x + 2 + \frac{2}{x-1}\,dx \\ &= x^2/2 + 2x + 2\ln|x-1| + C. \end{aligned}$$

(b) Compute $\int \frac{4+x}{(1+2x)(3-x)}\,dx =$
Solution:

$$\begin{aligned} \frac{4+x}{(1+2x)(3-x)} &= \frac{A}{1+2x} + \frac{B}{3-x} \\ 4 + x &= A(3-x) + B(1+2x) \\ 7 &= 7B \Rightarrow B = 1 \\ 4 &= 3A + 1 \Rightarrow A = 1 \\ \int \frac{4+x}{(1+2x)(3-x)} &= \int \frac{1}{3-x} + \frac{1}{1+2x}\,dx \\ &= -\ln|3-x| + \frac{1}{2}\ln|1+2x| + C. \end{aligned}$$

7. **Integration by Parts**

Use integration by parts to compute:

(a) $\int e^{-t} \cos(3t)\, dt =$

Solution:

$$\begin{aligned}
\int e^{-t} \cos(3t)\, dt &= \frac{1}{3}\sin(3t)e^{-t} - \int -e^{-t}\sin(3t)/e\, dt \\
&= \frac{1}{3}\sin(3t)e^{-t} + \frac{1}{3}\int e^{-t}\sin(3t)\, dt \\
\int e^{-t}\sin(3t)\, dt &= \frac{1}{3}e^{-t}(-\cos(3t)) - \int (-e^{-t})\frac{-1}{3}\cos(3t)\, dt \\
\int e^{-t}\cos(3t)\, dt &= \frac{1}{3}\sin(3t)e^{-t} - \frac{1}{9}\cos(3t)e^{-t} - \frac{1}{9}\int e^{-t}\cos(3t)\, dt \\
\frac{10}{9}\int e^{-t}\cos(3t)\, dt &= \frac{1}{3}\sin(3t)e^{-t} - \frac{1}{9}\cos(3t)e^{-t} + C \\
\int e^{-t}\cos(3t)\, dt &= \frac{3}{10}\sin(3t)e^{-t} - \frac{1}{10}\cos(3t)e^{-t} + C.
\end{aligned}$$

(b) $\int (x-3)\sin(\pi x)\, dx =$

Solution:

$$\begin{aligned}
\int (x-3)\sin(\pi x)\, dx &= \frac{-1}{\pi}\cos(\pi x)(x-3) - \int \frac{-1}{\pi}\cos(\pi x)\, dx \\
&= \frac{-1}{\pi}\cos(\pi x)(x-3) + \frac{1}{\pi}\int \cos(\pi x)\, dx \\
&= \frac{-1}{\pi}\cos(\pi x)(x-3) + \frac{1}{\pi^2}\sin(\pi x) + C.
\end{aligned}$$