Math 1232 Spring 2021
Single-Variable Calculus II Mastery Quiz 11
Due Friday, April 16

This week's mastery quiz has nine topics. You should do topics 22 and 21, and optionally *one* of the previous topics. Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 20-30 minutes on this quiz.

Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

22. Computing Taylor Series
21. Theory of Taylor Series
20. Power Series as Functions
19. Power Series
18. Absolute and Conditional Convergence
17. Comparison Test and Limit Comparison Test
16. Divergence and Integral tests
3. Derivatives of Exponentials and Logs
2. Exponents and Logarithms

22. **Computing Taylor Series**

(a) Using series we already know, write down a formula for the (infinite) Taylor series for x^3e^{2x}, and then write down the degree-six polynomial explicitly.

Solution:

We can take this from the series for e^x. So we have

$$\begin{aligned} e^x &= \sum_{n=0}^{\infty} \frac{x^n}{n!} \\ e^{2x} &= \sum_{n=0}^{\infty} \frac{2^n x^n}{n!} \\ x^3e^{2x} &= \sum_{n=0}^{\infty} \frac{2^n x^{n+3}}{n!} \\ T_6(x,0) &= x^3 + 2x^4 + \frac{4}{2}x^5 + \frac{8}{6}x^6 \end{aligned}$$

(b) Using series we already know, write down a formula for the (infinite) Taylor series for $(1-2x)^{-3}$, and then write down the degree-four polynomial explicitly.

Solution:

We can take this from the binomial series. So we have

$$\begin{aligned} f(x) &= \sum_{n=0}^{\infty} \binom{-3}{n}(-2x)^n = \sum_{n=0}^{\infty} \binom{-3}{n}(-2)^n x^n \\ T_4(x,0) &= 1 + (-2)\frac{-3}{1}x + 4\frac{12}{2}x^2 + (-8)\frac{-60}{6}x^3 + (16)\frac{360}{24}x^4 \\ &= 1 + 6x + 24x^2 + 80x^3 + 240x^4 \end{aligned}$$

21. **Theory of Taylor Series**

(a) Let $f(x) = \cos^2(x)$. Use *the definition of a Taylor series* to find $T_4(x, \pi)$ for this function. (That is, find the terms up through the cubic term.)

Solution:

$$\begin{aligned} f(x) &= \cos^2(x) & f(\pi) &= 1 \\ f'(x) &= -2\cos(x)\sin(x) & f'(\pi) &= 0 \\ f''(x) &= 2\sin^2(x) - 2\cos^2(x) & f''(\pi) &= -2 \\ f'''(x) &= 4\sin(x)\cos(x) + 4\cos(x)\sin(x) & f'''(\pi) &= 0 \\ f^{(4)} &= 8\cos^2(x) - 8\sin^2(x) & f^{(4)}(\pi) &= 8. \end{aligned}$$

So we have

$$T_4(x,\pi) = 1 - (x-\pi)^2 + \frac{1}{3}(x-\pi)^4.$$

(b) Using the Taylor series remainder, show that $\sin(x)$ is equal to its Maclaurin series.

Solution: We know that $\sin(x) = \sum_{n=1}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$. We know that $f^{n+1}(x) = \pm\cos(x)$ or $\pm\sin(x)$ so $|f^{n+1}(z)| \leq 1$, and thus

$$|R_n(x)| = \left|\frac{f^{(n+1)(z)}}{(n+1)!}x^{n+1}\right| \leq \frac{x^{n+1}}{(n+1)!}$$
$$0 \leq |R_n(x)| \leq \frac{x^{n+1}}{(n+1)!}$$

and we know that for any x, $\lim_{n\to\infty} x^{n+1}/(n+1)! = 0$. Thus, by the squeeze theorem, $\lim_{n\to\infty} R_n(x) = 0$ for any x, and thus $\sin(x) = T_{\sin}(x, 0)$ for any x.

20. **Power Series as Functions**

(a) Write a power series expression for $\frac{x^4}{2-4x}$ centered a 0. What is the radius of convergence?

Solution: We know that

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty}(2x)^n$$
$$\frac{x^4}{2}\frac{1}{1-2x} = \frac{x^4}{2}\sum_{n=0}^{\infty}(2x)^n$$
$$= \sum_{n=0}^{\infty}\frac{2^n}{2}x^{n+4}$$
$$\text{(or)} \quad = \sum_{n=4}^{\infty}2^{n+3}x^n.$$

The radius of convergence is 1/2. We can figure that out by reasoning from the geometric series: the radius of convergence for the geometric series is 1, so it converges for $-1 < 2x < 1$ or $-1/2 < x < 1/2$. Or we can use the ratio test:

$$\lim_{n\to\infty}\left|\frac{2^{n+4}x^{n+1}}{2^{n+3}x^n}\right| = \lim_{n\to\infty}|2x|$$

and thus it converges when $2|x| < 1$.

(b) If $f(x) = \sum_{n=0}^{\infty} 2^n n^3 (x-2)^n$, compute $\frac{d}{dx}f(x)$ and $\int f(x)\,dx$.

Solution:

$$\frac{d}{dx}f(x) = \sum_{n=0}^{\infty}2^n n^4 (x-2)^{n-1}$$
$$\text{(or better)} = \sum_{n=1}^{\infty}2^n n^4 (x-2)^{n-1}$$
$$\int f(x)\,dx = \sum_{n=0}^{\infty}\frac{2^n n^3}{n+1}(x-2)^{n+1} + C$$
$$\text{(or)} = \sum_{n=1}^{\infty}\frac{2^{n-1}(n-1)^3}{n}(x-2)^n + C.$$

19. **Power Series**

(a) Find the radius of convergence and the interval of convergence of $\sum_{n=0}^{\infty} \frac{(5x-3)^n}{\sqrt{n}}$.

Solution:

We use the ratio test.

$$\lim_{n\to\infty} \left| \frac{(5x-3)^{n+1}/\sqrt{n+1}}{(5x-3)^n/\sqrt{n}} \right| = \lim_{n\to\infty} \left| \frac{(5x-3)\sqrt{n}}{\sqrt{n+1}} \right|$$

$$= |5x-3| \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+1}} = |5x-3|.$$

So we need $|5x-3| < 1$ or $-1 < 5x-3 < 1$, or $2 < 5x < 4$ or $2/5 < x < 4/5$. We need to have x in the interval $(3/5 - 1/5, 3/5 + 1/5)$, so the radius is $1/5$.

To find the interval we need to check the endpoints. We see

$$\sum_{n=0}^{\infty} \frac{(2-3)^n}{\sqrt{n}} \sum_{n=0}^{\infty} \frac{(-1)^n}{\sqrt{n}}$$

converges by alternating series test

$$\sum_{n=0}^{\infty} \frac{(4-3)^n}{\sqrt{n}} = \sum_{n=0}^{\infty} \frac{1}{\sqrt{n}}$$

diverges by p-series test

Thus the interval of convergence is $[2/5, 4/5)$.

(b) Find the radius of convergence and the interval of convergence of $\sum_{n=0}^{\infty} \frac{n}{5^n}(x-3)^n$.

Solution:

We use the ratio test.

$$\lim_{n\to\infty} \left| \frac{(n+1)(x-3)^{n+1}/5^{n+1}}{(n)(x-3)^n/5^n} \right| = \lim_{n\to\infty} \left| \frac{n+1}{n} \frac{x-3}{5} \right|$$

$$= |x-3|/5 \lim_{n\to\infty} \frac{n+1}{n} = |x-3|/5.$$

So we need $|x-3|/5 < 1$ or $-5 < x-3 < 5$, or $-2 < x < 8$ or $3-5 < x < 3+5$. So the radius is 5.

To find the interval we need to check the endpoints. We see

$$\sum_{n=0}^{\infty} \frac{n}{5^n} 5^n = \sum_{n=0}^{\infty} n$$

diverges by divergence or p-series test

$$\sum_{n=0}^{\infty} \frac{n}{5^n} (-5)^n = \sum_{n=0}^{\infty} (-1)^n n$$

diverges by divergence test

Thus the interval is $(-2, 8)$.

18. **Absolute and Conditional Convergence**

For each series, tell whether it absolutely converges, conditionally converges, or diverges. Justify your answer (and in particular, if it conditionally converges, explain why it doesn't absolutely converge).

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

Solution: This clearly converges by the alternating series test, but does it absolutely converge? The ratio test won't work; if we work it out we'll get a limit of 1. But we have

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{n^2} \right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges by the p-series test, so our series converges absolutely.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3n+2}$

Solution: This series converges by the alternating series test. Again, we can't use the ratio test. But we can look at

$$\sum_{n=1}^{\infty} \left| \frac{(-1)^n}{3n+2} \right| = \sum_{n=1}^{\infty} \frac{1}{3n+2}$$

$$\lim_{n \to \infty} \frac{1/3n+2}{1/n} = \lim_{n \to \infty} \frac{n}{3n+2} = 1/3$$

is finite non-zero, so this series has the same convergence as $\sum \frac{1}{n}$, which diverges. Thus our original series converges conditionally.

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{3^n+2}$

Solution:

We use the ratio test. We have

$$\lim_{n \to \infty} \left| \frac{(-1)^{n+1}/3^{n+1}+2}{(-1)^n/3^n+2} \right| = \lim_{n \to \infty} \frac{3^n+2}{3^{n+1}+2} = \frac{1}{3}.$$

So by the ratio test this series converges absolutely.

17. **Comparison Tests**

Determine whether each of the following series converges by using an appropriate comparison test.

(a) $\sum_{n=1}^{\infty} \frac{n^2+e^{-n}}{n^3-\ln(n^3+1)}$

Solution:

You can't really use the limit comparison test here, at least not easily, because the expression is complicated. But you can use the usual comparison test. We know that $n^2 + e^{-n} > n^2$, and $n^3 - \ln(n^3+1) < n^3$, so

$$\frac{n^2 + e^{-n}}{n^3 - \ln(n^3+1)} > \frac{n^2}{n^3} = \frac{1}{n}.$$

We know that $\sum \frac{1}{n}$ diverges by the p-series test, so our series diverges by the comparison test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2 + \sqrt{n} + 2}{n^4 - n^{4/3} + 2}$

Solution:

Here it would be hard to use the regular comparison test. It's true that $\frac{n^2+\sqrt{n}+2}{n^4-n^{4/3}+2} \geq \frac{1}{n^2}$, but since this says it's greater than a convergent series, it doesn't really help. Instead, we limit compare to $\frac{1}{n^2}$. We have

$$\lim_{n\to\infty} \frac{n^2+\sqrt{n}+2 / n^4 - n^{4/3} + 2}{1/n^2} = \lim_{n\to\infty} \frac{n^4 + n^{5/2} + 2n^2}{n^4 - n^{4/3} + 2} = 1.$$

Since this is a finite non-zero number, the two series have the same convergence behavior. Thus, since $\frac{1}{n^2}$ converges, we know that our series also converges.

16. **Divergence and Integral Tests**

Determine whether each of the following series converges or diverges. Justify your answers using only the divergence and integral tests (and *not* the comparison tests or ratio test or root test).

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n\ln(n)}{\ln(\ln(\ln(n)))}$

Solution:

I don't want to think about integrating this (though it's too late now, I already have). But I don't have to. We know the limit of $\frac{n\ln(n)}{\ln(\ln(\ln(n)))}$ is ∞, and since the terms don't go to zero, the series diverges by the divergence test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^4+7}$

Solution: We have

$$\begin{aligned}\int_1^{\infty} \frac{x^3}{x^4+7}\,dx &= \lim_{t\to\infty} \int_1^t \frac{x^3}{x^4+7}\,dx \\ &= \lim_{t\to\infty} \frac{1}{4} \ln(x^4+7)\Big|_1^t \\ &= \lim_{t\to\infty} \frac{1}{4}\big(\ln(t^4+7) - \ln(8)\big) = \infty.\end{aligned}$$

Thus by the integral test, $\sum_{n=1}^{\infty} \frac{n^3}{n^4+7}$ diverges.

(c) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2+1}$

Solution: We can work this out with the integral test. We have

$$\int_1^\infty xe^{-x^2+1}\,dx = \lim_{t\to\infty}\int_1^t xe^{-x^2+1}\,dx \quad = \lim_{t\to\infty}\frac{-1}{2}e^{-x^2+1}\Big|_1^t = \lim_{t\to\infty}\frac{1}{2}e^2 - \frac{1}{2}e^{-t^2+1} = e^2/2 < \infty.$$

Since this integral converges, the series must also converge by the integral test.

3. **Derivatives of Exponentials and Logarithms**

(a) Compute $\dfrac{d}{dx}x^{\sqrt{x^2+1}}$

Solution:

$$\begin{aligned} y &= x^{\sqrt{x^2+1}} \\ \ln|y| &= \sqrt{x^2+1}\ln|x| \\ y'/y &= \frac{x\ln|x|}{\sqrt{x^2+1}} + \frac{\sqrt{x}}{x} \\ y' &= x^{\sqrt{x^2+1}}\left(\frac{x\ln|x|}{\sqrt{x^2+1}} + \frac{\sqrt{x}}{x}\right) \end{aligned}$$

(b) Find an equation for the tangent line to the curve $y = \frac{e^x}{x^2}$ at the point $(1, e)$.

Solution: We have $y' = \frac{e^x x^2 - 2xe^x}{x^4}$, and thus $y'(1) = -e$. Then the equation of the tangent line is

$$y - e = -e(x-1) = e - ex$$

or

$$y = -e(x-1) + e = 2e - ex.$$

(c) Compute $\dfrac{d}{dx}\cos(\ln|x-3|)$.

Solution:

$$\frac{d}{dx}\cos(\ln|x-3|) = -\sin(\ln|x-3|)\cdot\frac{1}{x-3}.$$

2. **Topic 2: Exponents and Logarithms**

(a) Showing your work, compute $\log_3(15)+2\log_3(6)-\log_3(20)$. (Give an exact answer with no decimals.)

Solution:

$$\log_3(15) + 2\log_3(6) - \log_3(20) = \log_3(15/20) + \log_3(36) = \log_3(27) = 3.$$

(b) Give an exact solution for the equation $e^{3x^2-2} = 4$.

Solution:

$$\begin{aligned} 3x^2 - 2 &= \ln(4) \\ 3x^2 &= \ln(4) + 2 \\ x^2 &= \frac{\ln(4)+2}{3} \\ x &= \pm\sqrt{\frac{\ln(4)+2}{3}}. \end{aligned}$$

(Technically giving just the positive solution is fully correct here, since I just asked for "an" exact solution. I'm interested in whether you can do the exponent and log manipulations, not whether you remember the $\pm$ on the square root, here.)

(c) Compute $4^{\log_2(11) - 3log_2(3)}$. (Give an exact answer with no decimals.)

Solution:

$$\begin{aligned} 4^{\log_2(11) - 3\log_2(3)} &= \frac{2^{2\log_2(11)}}{2^{6\log_2(3)}} \\ &= \frac{11^2}{3^6} = \frac{121}{729}. \end{aligned}$$

(I'd be happy to just accept $11^2/3^6$ here.)

(d) Give an exact solution for the equation $\log_5(7x - 24) = 2$.

Solution:

$$\begin{aligned} 7x - 24 &= 25 \\ 7x &= 49 \\ x &= 7. \end{aligned}$$