Math 1232 Spring 2021
Single-Variable Calculus II Mastery Quiz 12
Due Friday, April 23

This week's mastery quiz has fourteen topics. You should do topics 24 and 23, and optionally *one* of the previous topics. Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 20-30 minutes on this quiz.

Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

24. Parametrization
23. Applications of Taylor Series
22. Computing Taylor Series
21. Theory of Taylor Series
20. Power Series as Functions
19. Power Series
18. Absolute and Conditional Convergence
17. Comparison Test and Limit Comparison Test
15. Geometric and Telescoping Series
13. Separable Differential Equations
11. Improper Integrals
10. Numeric Integration
6. L'Hospital's Rule
4. Integrals involving Exponentials and Logs

24. **Parametrization**

(a) Find a parametrization of the ellipse $x^2 + y^2/4 = 1$. (Hint: what are the x and y intercepts?)

Solution:

An ellipse is like a circle, but it's wider in one direction than the other. In particular, this ellipse goes through $(1,0), (0,2), (-1,0), (0,-2)$. So we can take

$$x = \cos(t)$$
$$y = 2\sin(t).$$

There are also lots of other options; another would be

$$x = \sin(t)$$
$$y = 2\cos(t).$$

This would start at a different point and go clockwise instead of counterclockwise, but still cover the entire ellipse.

(b) Find an equation of the line tangent to the curve $x = 1 + \sqrt{t}, y = t^3$ at the point $(2,1)$.

Solution:

We have $x'(t) = \frac{1}{2\sqrt{t}}$ and $y'(t) = 3t^2$. This point happens at $t = 1$, so we have $x'(t) = \frac{1}{2}$ and $y'(1) = 3$. Then

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{3}{1/2} = 6$$
$$y - 1 = 6(x - 2).$$

(c) Find the length of the curve parametrized by $x = e^t - t, y = 4e^{t/2}$ for $0 \le t \le 2$.

Solution:

We have $x'(t) = e^t - 1$ and $y'(t) = 2e^{t/2}$, so the arc length is

$$\begin{aligned} L &= \int_0^2 \sqrt{(x'(t))^2 + (y'(t))^2}\,dt \\ &= \int_0^2 \sqrt{(e^t - 1)^2 + (2e^{t/2})^2}\,dt \\ &= \int_0^2 \sqrt{e^{2t} - 2e^2 + 1 + 4e^t}\,dt = \int_0^2 \sqrt{e^{2t} + 2e^t + 1}\,dt \\ &= \int_0^2 e^t + 1\,dt = e^t + t\Big|_0^2 = e^2 + 2 - 1 = e^2 + 1. \end{aligned}$$

23. **Applications of Taylor Series**

(a) Use a Taylor series to compute $\lim_{x \to 0} \frac{\cos(x^2) - 1 + x^4/2}{x^8} =$

Solution:

$$\begin{aligned}\lim_{x\to 0}\frac{\cos(x^2)-1+x^4/2}{x^8} &= \lim_{x\to 0}\frac{(1-x^4/2+x^8/4!-x^{12}/6!+\dots)-1+x^4/2}{x^8}\\ &= \lim_{x\to 0}\frac{x^8/4!-x^{12}/6!+\dots}{x^8}\\ &= \lim_{x\to 0}\frac{1}{4!}-\frac{x^4}{6!}+\dots=\frac{1}{24}.\end{aligned}$$

(b) Use a degree-three Taylor polynomial to estimate $\sqrt{1.2}$.

Solution:

$$\begin{aligned}\sqrt{1+x} &= 1+\frac{1}{2}x+\frac{(1/2)(-1/2)}{2!}x^2+\frac{(1/2)(-1/2)(-3/2)}{3!}x^3\\ &= 1+\frac{x}{2}-\frac{x^2}{8}+\frac{x^3}{16}\\ \sqrt{1.2} &= 1+\frac{.2}{2}-\frac{.04}{8}+\frac{.008}{16}=1+.1-.005+.0005=1.0955.\end{aligned}$$

(c) Use a degree-five Taylor polynomial to estimate arctan(.1).

Solution:

We have

$$\begin{aligned}\arctan(x) &\approx x-x^3/3+x^5/5\\ \arctan(.1) &\approx .1-(.1)^3/3+(.1)^5/5=.1-.00033\dots+.000002=.09966866\dots.\end{aligned}$$

22. **Computing Taylor Series**

(a) Using series we already know, write down a formula for the (infinite) Taylor series for $x^2\ln(1-2x^3)$, and then write down the degree-eleven polynomial explicitly.

Solution:

We can take this from the series for $\ln(1+x)$. So we have

$$\begin{aligned}\ln(1+x) &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{x^n}{n}\\ \ln(1-2x^3) &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-2x^3)^n}{n}\\ x^2\ln(1-2x^3) &= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(-1)^n2^nx^{3n+2}}{n}\\ &= \sum_{n=1}^{\infty}\frac{-2^n}{n}x^{3n+2}T_{11}(x,0) \qquad = -2x^5-2x^8-\frac{8}{3}x^{11}.\end{aligned}$$

(b) Using series we already know, write down a formula for the (infinite) Taylor series for $(1+3x)^{2/3}$, and then write down the degree-three polynomial explicitly.

Solution:

We can take this from the binomial series. So we have

$$f(x) = \sum_{n=0}^{\infty} \binom{2/3}{n} (3x)^n = \sum_{n=0}^{\infty} \binom{2/3}{n} (3)^n x^n$$

$$T_3(x,0) = 1 + \frac{2/3}{1} \cdot 3x + \frac{(2/3)(-1/3)}{2} \cdot 3^2 x^2 + \frac{(2/3)(-1/3)(-4/3)}{6} \cdot 3^3 x^3$$

$$= 1 + 2x - x^2 + \frac{4}{3} x^3.$$

21. **Theory of Taylor Series**

(a) Let $f(x) = e^{x^2}$. Use *the definition of a Taylor series* to find $T_4(x,0)$ for this function. (That is, find the terms up through the degree-four term.)

Solution:

$$f(x) = e^{x^2} \qquad f(0) = 1$$

$$f'(x) = 2xe^{x^2} \qquad f'(0) = 0$$

$$f''(x) = 2e^{x^2} + 4x^2 e^{x^2} \qquad f''(0) = 2 f'''(x) \quad = 4xe^{x^2} + 8xe^{x^2} + 8x^3 e^{x}$$

$$f^{(4)} = 12e^{x^2} + 24x^2 e^{x^2} + 24x^2 e^{x^2} + 16x^4 e^{x^2} \quad f^{(4)}(0) = 12.$$

So we have

$$T_4(x,\pi) = 1 + \frac{2}{2} x^2 + \frac{12}{4!} x^4 = 1 + x^2 + x^4/2.$$

(b) Estimate the error you useif $T_3(x) = x - \frac{x^2}{2} + \frac{x^3}{3}$ to approximate $g(x) = \ln(1+x)$ at $x = .5$.

Solution: We can use the remainder theorem. We have

$$|R_3(x)| = \left| \frac{f^{(4)}(z)}{4!} x^4 \right| = \left| \frac{3!}{(1+z)^4 4!} x^4 \right| = \left| \frac{x^4}{4(1+z)^4} \right|$$

We know that z is between 0 and .5, so we have

$$\frac{1}{(1+z)^4} \leq \frac{1}{1^4} = 1$$

$$|R_3(.5)| = \left| \frac{.5^4}{4(1+z)^4} \right| \leq \frac{.5^4}{4} \cdot 1 = \frac{1}{64}.$$

20. **Power Series as Functions**

(a) Write a power series expression for $\frac{2x^3}{5-x}$ centered a 0. What is the radius of convergence?

Solution: We know that

$$\frac{1}{5-x} = \frac{1}{5}\frac{1}{1-x/5} = \frac{1}{5}\sum_{n=0}^{\infty}(x/5)^n$$

$$\frac{2x^3}{2}\frac{1}{5-x} = \frac{2x^3}{5}\sum_{n=0}^{\infty}(x/5)^n$$

$$= \sum_{n=0}^{\infty}\frac{2}{5^{n+1}}x^{n+3}$$

$$\text{(or)} \quad = \sum_{n=3}^{\infty}\frac{2}{5^{n-2}}x^n.$$

The radius of convergence is 5. We can figure that out by reasoning from the geometric series: the radius of convergence for the geometric series is 1, so it converges for $-1 < x/5 < 1$ or $-5 < x < 5$. Or we can use the ratio test:

$$\lim_{n\to\infty}\left|\frac{2x^{n+1}/5^{n-1}}{2x^n/5^{n-2}}\right| = \lim_{n\to\infty}|x/5|$$

and thus it converges when $|x|/5 < 1$.

(b) If $f(x) = \sum_{n=0}^{\infty}\frac{3^n}{n!}(x+2)^n$, compute $\frac{d}{dx}f(x)$ and $\int f(x)\,dx$.

Solution:

$$\frac{d}{dx}f(x) = \sum_{n=0}^{\infty}\frac{3^n}{(n-1)!}(x+2)^{n-1}\text{(or much much better)} \quad = \sum_{n=1}^{\infty}\frac{3^n}{(n-1)!}(x+2)^{n-1}$$

$$\int f(x)\,dx = \sum_{n=0}^{\infty}\frac{3^n}{(n+1)!}(x+2)^{n+1} + C$$

$$\text{(or)} = \sum_{n=1}^{\infty}\frac{3^{n-1}}{n!}(x+2)^n + C.$$

19. **Power Series**

(a) Find the radius of convergence and the interval of convergence of $\sum_{n=0}^{\infty}(n(x-3))^n$.

Solution:
We use the ratio test.

$$\lim_{n\to\infty}\left|\frac{(n+1)^{n+1}(x-3)^{n+1}}{n^n(x-3)^n}\right| = \lim_{n\to\infty}|x-3|\left(\frac{n+1}{n}\right)^n(n+1) \quad \geq |x-3|\lim_{n\to\infty}(n+1) = \infty$$

unless $x = 3$. So the radius of convergence is 0, and the series converges if and only if $x = 3$.

(b) Find the radius of convergence and the interval of convergence of $\sum_{n=0}^{\infty}\frac{2^n}{n^2+n}x^n$.

Solution:
We use the ratio test.

$$\lim_{n\to\infty}\left|\frac{2^{n+1}x^{n+1}/(n+1)^2+n+1}{2^nx^n/n^2+n}\right| = \lim_{n\to\infty} 2|x|\frac{n^2+3n+2}{n^2+n}$$
$$= 2|x|.$$

So we need $2|x| < 1$ or $-1 < 2x < 1$, or $-1/2 < x < 1/2$ We need to have x in the interval $(0-1/2, 0+1/2)$, so the radius is $1/2$.
To find the interval we need to check the endpoints. We see

$$\sum_{n=0}^{\infty}\frac{2^n}{n^2+n}(1/2)^n \sum_{n=0}^{\infty}\frac{1}{n^2+n}$$

converges by comparison to a p-series

$$\sum_{n=0}^{\infty}\frac{2^n}{n^2+n}(-1/2)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{n^2+n}$$

converges by the alternating series test

Thus the interval of convergence is $[-1/2, 1/2]$.

18. **Absolute and Conditional Convergence**

For each series, tell whether it absolutely converges, conditionally converges, or diverges. Justify your answer (and in particular, if it conditionally converges, explain why it doesn't absolutely converge).

(a) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{3^n}$

Solution: This converges absolutely by the ratio test. We compute

$$\lim_{n=1}^{\infty}\frac{(n+1)/3^{n+1}}{n/3^n} = \lim_{n\to\infty}\frac{(n+1)3^n}{n3^{n+1}} = \lim_{n\to\infty}\frac{n+1}{3n} = \frac{1}{3} < 1.$$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{5^n}{5^n+1}$

Solution: This series diverges by the divergence test. We can see the terms have the limit

$$\lim_{n\to\infty}\frac{5^n}{5^n+1} = \lim_{n\to\infty}\frac{1}{1+1/5^n} = 1$$

which is non-zero, so the series diverges.

(c) $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{n2^n+1}$

Solution: You might try the ratio test here, but it won't actually help:

$$\lim_{n\to\infty}\left|\frac{(-2)^{n+1}/(n+1)2^{n+1}+1}{(-2)^n/n2^n+1}\right| = \lim_{n\to\infty}\frac{2(n2^n+1)}{(n+1)2^{n+1}+1} = \lim_{n\to\infty}\frac{n+1/2^{n+1}}{n+1+1/2^{n+1}} = 1.$$

Instead, we observe that this is an alternating series with the terms tending to zero, since

$$\lim_{n\to\infty} \frac{(-2)^n}{n2^n+1} = \lim_{n\to\infty} \frac{(-1)^n}{n+1/2^n} = 0.$$

Thus it converges. However, if we look at the absolute value, we can compare it to the series $\sum \frac{1}{n}$:

$$\lim_{n\to\infty} \frac{2^n/n2^n+1}{1/n} = \lim_{n\to\infty} \frac{n2^n}{n2^n+1} = 1$$

and since $\sum \frac{1}{n}$ diverges, by the limit comparison test our absolute-value series also diverges. Thus the original series converges conditionally.

17. **Comparison Tests**

Determine whether each of the following series converges by using an appropriate comparison test.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{3n^3-2n^2+n}{4n^6+4n^3-n}$

Solution:

Here it would be hard to use the regular comparison test. Instead, we limit compare to $\frac{1}{n^3}$. We have

$$\lim_{n\to\infty} \frac{3n^3-2n^2+n/4n^6+4n^3-n}{1/n^3} = \lim_{n\to\infty} \frac{3n^6-2n^5+n^4}{4n^6+4n^3-n} = 3/4.$$

Since this is a finite non-zero number, the two series have the same convergence behavior. Thus, since $\frac{1}{n^3}$ converges, we know that our series also converges.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^3}{n^4\cos^2(n^2)-n}$

Solution:

You can't really use the limit comparison test here, because the obvious comparsion $\sum 1/n$ leads to you needing to compute $\lim_{n\to\infty}\cos^2(n^2)$, which doesn't exist.

But you can use the usual comparison test. We know that

$$\begin{aligned} n^4\cos^2(n^2)-n &\le n^4 \\ \frac{1}{n^4\cos^2(n^2)-n} &\ge \frac{1}{n^4} \\ \frac{n^3}{n^4\cos^2(n^2)-n} &\ge \frac{n^3}{n^4} = \frac{1}{n}. \end{aligned}$$

Since $\sum 1/n$ diverges, by the comparison test our series also diverges.

15. **Geometric and Telescoping Series**

Compute the following infinite sums, with justification:

(a) $\sum_{n=1}^{\infty} \frac{5^{n-3}}{2^n} =$

Solution:

This is a geometric series with $a = \frac{1/25}{2} = \frac{1}{50}$ and $r = \frac{5}{2}$. Since $5/2 > 1$, this sum diverges.

(b) $\sum_{n=1}^{\infty} \frac{(-4)^{n-1}}{5^n} =$

Solution:

This is a geometric series with $a = \frac{1}{5}$ and $r = \frac{-4}{5}$. Since $|r| = 4/5 < 1$ this series converges, to $\frac{1/5}{1+4/5} = \frac{1}{9}$.

(c) $\sum_{n=1}^{\infty} \frac{-1}{n^2 + 5n + 6} =$

Solution: This is the same as

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n+3} - \frac{1}{n+2} &= \lim_{t\to\infty} \sum_{n=1}^{t} \frac{1}{n+3} - \frac{1}{n+2} \\
&= \lim_{t\to\infty} \left((1/4 - 1/3) + (1/5 - 1/4) + \dots \right) \\
&= \lim_{t\to\infty} \frac{1}{t+3} - \frac{1}{3} = \frac{-1}{3}.
\end{aligned}$$

Thus this sum diverges to infinity.

13. **Differential Equations**

(a) Find a general solution to the equation $y' = e^{x+y}$.

Solution:

$$\begin{aligned}
\frac{dy}{dx} &= e^x e^y \\
e^{-y}\,dy &= e^x\,dx \\
-e^{-y} &= e^x + C \\
-y &= \ln(-e^x - C) \\
y &= -\ln(-e^x - C).
\end{aligned}$$

(b) Find a (specific) solution to the initial value problem $x + y\sqrt{x^2+1}y' = 0$ with $y(0) = 2$.

Solution:

$$\begin{aligned}
y\sqrt{x^2+1}\frac{dy}{dx} &= -x \\
y\,dy &= \frac{-x}{\sqrt{x^2+1}}\,dx \\
y^2/2 &= -\sqrt{x^2+1} + C \\
y &= \sqrt{2C - 2\sqrt{x^2+1}}
\end{aligned}$$

Then we have

$$\begin{aligned}
2 &= \sqrt{2C - 2\sqrt{0+1}} = \sqrt{2C-2} \\
4 &= 2C - 2 \\
C &= 3 \\
y &= \sqrt{6 - 2\sqrt{x^2+1}}.
\end{aligned}$$

11. **Improper Integrals**

(a) Compute $\displaystyle\int_0^7 \frac{1}{\sqrt[3]{7-x}}\,dx$.

Solution: We know that $\frac{1}{\sqrt[3]{7-x}}$ is undefined at 7. So we have

$$\begin{aligned}
\int_0^7 \frac{1}{\sqrt[3]{7-x}}\,dx &= \lim_{t\to\pi/2} \int_0^t \frac{1}{\sqrt[3]{7-x}}\,dx \\
&= \lim_{t\to\pi/2} -\frac{3}{2}(7-x)^{2/3} \\
&= \lim_{t\to\pi/2} -0 + \frac{3}{2}(7-t)^{2/3} = \frac{3}{2}7^{2/3}.
\end{aligned}$$

(b) Compute $\displaystyle\int_{-\infty}^{\infty} xe^{-x^2}\,dx$.

Solution: We'll take $u = -x^2$ so $du = -2x\,dx$ and $xe^{-x^2} = \frac{-1}{2}e^u$. Then

$$\begin{aligned}
\int_{-\infty}^{\infty} xe^{-x^2}\,dx &= \lim_{t\to\infty} \int_{-\infty}^{t} xe^{-x^2}\,dx \\
&= \lim_{t\to\infty}\lim_{s\to-\infty} \int_s^t xe^{-x^2}\,dx \\
&= \lim_{t\to\infty}\lim_{s\to-\infty} \int_{-s^2}^{-t^2} -\frac{1}{2}e^u\,du \\
&= \lim_{t\to\infty}\lim_{s\to\infty} -\frac{1}{2}e^u\Big|_{-s^2}^{-t^2} = \lim_{t\to\infty}\lim_{s\to\infty} \frac{1}{2}e^{-s^2} - \frac{1}{2}e^{-t^2} \\
&= 0 - 0 = 0.
\end{aligned}$$

10. **Numeric Integration**

(a) How many intervals do you need with the **trapezoid** rule to approximate $\int_2^5 \frac{x}{x+1}$ to within 1/200? Compute that approximation. (Feel free to use a calculator to plug in numeric values, or to leave the answer in exact unsimplified terms, but show every step.)

Solution:

We have

$$
\begin{aligned}
f'(x) &= \frac{(x+1)-x}{(x+1)^2} = \frac{1}{(x+1)^2} \\
f''(x) &= \frac{-2}{(x+1)^3} \\
|f''(x)| &\le \frac{2}{3^3} = \frac{2}{27}.
\end{aligned}
$$

Then we have

$$|E_T| \le \frac{2/27 \cdot 3^3}{12n^2} = \frac{1}{6n^2}.$$

We want $1/200 \ge \frac{1}{6n^2}$ which implies

$$
\begin{aligned}
n^2 &\ge \frac{200}{6} = 100/3 \approx 33 \\
n &\ge 6.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
T_6 &= \frac{1}{4}f(2) + \frac{1}{2}f(2.5) + \frac{1}{2}f(3) + \frac{1}{2}f(3.5) + \frac{1}{2}f(4) + \frac{1}{2}f(4.5) + \frac{1}{4}f(5) \\
&= \frac{1}{6} + \frac{5}{14} + \frac{3}{8} + \frac{7}{18} + \frac{4}{10} + \frac{9}{22} + \frac{5}{24} \\
&= \frac{31949}{13860} \approx 2.30512.
\end{aligned}
$$

Since the true answer is $3 - \ln(2) \approx 2.30685$ this is in fact within our margin of error.

(b) Suppose we have

$$g(0) = 5 \quad g(1) = 4 \quad g(2) = 2 \quad g(3) = 3 \quad g(4) = 5 \quad g(5) = 6 \quad g(6) = 5$$

Approximate $\int_0^6 g(x)\,dx$ using the midpoint rule and the Simpson's rule.

Solution:

For the midpoint rule, we have

$$T_3 = 2g(1) + 2g(3) + 2g(5) = 8 + 6 + 12 = 26.$$

For the Simpson's rule, we have

$$S_6 = \frac{1}{3}(5 + 16 + 8 + 12 + 10 + 24 + 5) = \frac{80}{3} \approx 26.67.$$

6. **L'Hospital's Rule**

Compute the following limits:

(a) $\lim_{x\to 1} \frac{x^7-1)}{x^4-1} =$

Solution:

$$\lim_{x\to 1} \frac{x^7-1}{x^4-1} = \lim_{x\to 1} \frac{7x^6}{4x^3} = 7/4.$$

(b) $\lim_{x\to\infty} \sqrt{x}e^{-x} =$

Solution:

$$\begin{aligned}\lim_{x\to\infty} \sqrt{x}e^{-x} &= \lim_{x\to\infty} \frac{\sqrt{x}}{e^x} = \lim_{x\to\infty} \frac{1/2\sqrt{x}}{e^x} \\ &= \lim_{x\to\infty} \frac{1}{2e^x\sqrt{x}} = 0.\end{aligned}$$

(c) $\lim_{x\to 0}(1-3x)^{1/x} =$

Solution:

$$\begin{aligned}\ln(y) &= \frac{1}{x}\ln(1-3x) \\ \lim_{x\to 0}\ln(y) &= \lim_{x\to 0} \frac{\ln(1-3x)}{x} = \lim_{x\to 0} \frac{-3/1-3x}{1} = -3 \\ \lim_{x\to 0} y &= e^{-3} = \frac{1}{e^3}.\end{aligned}$$

4. **Integrals Involving Exponentials and Logarithms**

Compute the following integrals:

(a) $\int_1^2 \frac{e^{1/x}}{x^2}\,dx =$

Solution: We take $u = 1/x$ so $du = -1/x^2\,dx$. Then

$$\begin{aligned}\int_1^2 \frac{e^{1/x}}{x^2}\,dx &= \int_1^{1/2} -e^u\,du \\ &= -e^u\Big|_1^{1/2} = \sqrt{e} - e.\end{aligned}$$

(b) $\int \frac{1+2x+x^2}{3x+3x^2+x^3}\,dx =$

Solution:

Take $u = 3x+3x^2+x^3$ so $du = 3+6x+3x^2\,dx$. Then

$$\int \frac{1+2x+x^2}{3x+3x^2+x^3}\,dx = \int \frac{du}{3u} = \ln|u|/3 + C = \frac{1}{3}\ln|3x+3x^2+x^3| + C.$$

(c) $\displaystyle\int \frac{2^x}{2^x+3}\,dx =$

Solution:

We can take $u = 2^x + 3$ so that $du = 2^x \ln(2)\,dx$. Then

$$\begin{aligned}\int \frac{2^x}{2^x+3}\,dx &= \int \frac{1}{\ln(2)u}\,du \\ &= \frac{\ln|u|}{\ln(2)} + C = \frac{\ln|2^x+3|}{\ln(2)} + C.\end{aligned}$$