Math 1232 Spring 2021
Single-Variable Calculus II Mastery Quiz 13
Due Friday, April 30

This week's mastery quiz has thirteen topics. You can submit up to three topics. Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 20-30 minutes on this quiz.

Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

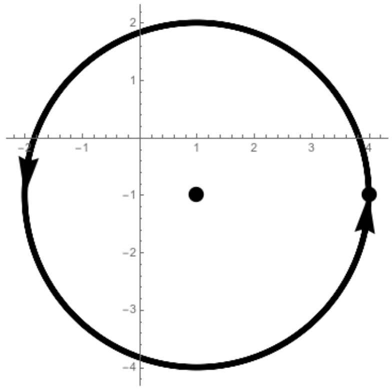
Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

24. Parametrization
23. Applications of Taylor Series
22. Computing Taylor Series
21. Theory of Taylor Series
20. Power Series as Functions
19. Power Series
18. Absolute and Conditional Convergence
17. Comparison Test and Limit Comparison Test
15. Geometric and Telescoping Series
14. Sequences
12. Arc Length and Surface Area
11. Improper Integrals
10. Numeric Integration
4. Integrals involving exponents and logs
1. Inverse Functions

24. **Parametrization**

(a) Find a parametrization for the circle of radius 3 centered at $(1, -1)$, starting at $(4, -1)$ and going counterclockwise twice around the circle.



Solution:

A circle has parametrization $\vec{r}(t) = (\cos(t), \sin(t))$. To make it radius 3 we multiply by 3, and then we shift it over to have center $(1, -1)$, and get

$$\vec{r}(t) = 3\cos(t) + 1, 3\sin(t) - 1.$$

In order to make it go around twice, we have $0 \leq t \leq 4\pi$. Alternatively, we could have $0 \leq 2 \leq 2\pi$ and use the equations

$$\vec{r}(t) = 3\cos(2t) + 1, 3\sin(2t) - 1.$$

(There are a bunch of other options that also work but these are the two most obvious to me.)

(b) Find an equation of the line tangent to the curve $x = \cos^3(t), y = \sin^3(t)$ at the point $(1/8, -3\sqrt{3}/8)$.

Solution:

We have $x'(t) = -3\cos^2(t)\sin(t)$ and $y'(t) = 3\sin^2(t)\cos(t)$. This point happens at $t = -\pi/3$, so we have

$$\begin{aligned} x'(-\pi/3) &= 3\sqrt{3}/8 \\ y'(-\pi/3) &= 9/8 \\ \frac{dy}{dx} &= \frac{dy/dt}{dx/dt} = \frac{9/8}{3\sqrt{3}/8} = \sqrt{3} \\ y + \frac{3\sqrt{3}}{8} &= \sqrt{3}(x - 1/8). \end{aligned}$$

(c) Find the length of the curve parametrized by $x = 3t^2, y = 3t - t^3$, for $1 \leq t \leq 4$.

Solution:

We have $x'(t) = 6t$ and $y'(t) = 3 - 3t^2$, so the arc length is

$$\begin{aligned} L &= \int_1^4 \sqrt{(x'(t))^2 + (y'(t))^2}\, dt \\ &= \int_1^4 \sqrt{(6t)^2 + (3 - 3t^2)^2}\, dt \\ &= \int_1^4 \sqrt{36t^2 + 9 - 18t^2 + 9t^4}\, dt \\ &= \int_1^4 3\sqrt{1 + 2t^2 + t^4}\, dt \\ &= \int_1^4 3 + 3t^2\, dt = 3t + t^3\Big|_1^4 = 12 + 64 - 3 - 1 = 72. \end{aligned}$$

23. **Applications of Taylor Series**

(a) Use a Taylor series to compute $\lim_{x\to 0} \dfrac{xe^{x^3} - x - x^4}{x^7} =$

Solution:

$$\begin{aligned} \lim_{x\to 0} \frac{xe^{x^3} - x - x^4}{x^7} &= \lim_{x\to 0} \frac{(x + x^4 + x^7/2 + x^{10}/3! + \dots) - x + x^4}{x^7} \\ &= \lim_{x\to 0} \frac{x^7/2 + x^{10}/3! + \dots}{x^7} \\ &= \lim_{x\to 0} \frac{1}{2} - \frac{x^3}{3!} + \dots = \frac{1}{2}. \end{aligned}$$

(b) Use a degree-three Taylor polynomial to estimate $(1.1)^{3.1}$.

Solution:

$$\begin{aligned} (1.1)^{3.1} &\approx 1 + 3.1x + \frac{3.1 \cdot 2.1}{1 \cdot 2}x^2 + \frac{3.1 \cdot 2.1 \cdot 1.1}{1 \cdot 2 \cdot 3}x^3 \\ &= 1 + 3.1x + 3.255x^2 + 1.1935x^3 \\ (1.1)^{3.1} &\approx 1 + 3.1(.1) + 3.255(.1)^2 + 1.1935(.1)^3 = 1 + .31 + .03255 + .0011935 = 1.3437435. \end{aligned}$$

(c) Use a degree-five Taylor polynomial to estimate $\sin(.3)$.

Solution:

We have

$$\begin{aligned} \sin(x) &\approx x - x^3/6 + x^5/120 \\ \sin(.3) &\approx .3 - .3^3/6 + .3^5/120 \approx .29552. \end{aligned}$$

22. **Computing Taylor Series**

(a) Using series we already know, write down a formula for the (infinite) Taylor series for $x(8 + x)^{5/3}$, and then write down the degree-four polynomial explicitly.

Solution:

We can take this from the binomial series for $(1+x)^\alpha$. So we have

$$(1+x/8)^{5/3} = \sum_{n=0}^{\infty} \binom{5/3}{n} (x/8)^n$$

$$(8+x)^{5/3} = 32(1+x/8)^{5/3} = \sum_{n=0}^{\infty} \binom{5/3}{n} 3^{5-3n} x^n$$

$$x(8+x)^{5/3} = \sum_{n=0}^{\infty} \binom{5/3}{n} 3^{5-3n} x^{n+1}$$

$$T_4(x,0) = 32x + {}^5\!/_3 \cdot 4x^2 + \frac{({}^5\!/_3)({}^2\!/_3)}{2} \cdot \frac{1}{2}x^3 + \frac{({}^5\!/_3)({}^2\!/_3)(-{}^1\!/_3)}{6}\frac{1}{16}x^4$$

$$= 32x + \frac{20}{3}x^2 + \frac{5}{18}x^3 - \frac{5}{1296}x^4.$$

(b) Using series we already know, write down a formula for the (infinite) Taylor series for $e^{3x} - e^x$, and then write down the degree-three polynomial explicitly.

Solution:

We can take this from the known series for e^x. So we have

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$e^{3x} = \sum_{n=0}^{\infty} \frac{(3x)^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n}{n!}x^n$$

$$e^{3x} - e^x = \sum_{n=0}^{\infty} \frac{3^n - 1}{n!}x^n$$

$$T_3(x,0) = 0 + \frac{2}{1}x + \frac{8}{2}x^2 + \frac{26}{6}x^3$$

$$= x + 4x^2 + \frac{13}{3}x^3.$$

21. **Theory of Taylor Series**

(a) Let $f(x) = \sin(x)$. Use *the definition of a Taylor series* to find $T_3(x, \pi/6)$ (centered at $\pi/6$) for this function. (That is, find the terms up through the degree-three term.)

Solution:

$$f(x) = \sin(x) \qquad f(\pi/4) = 1/2$$

$$f'(x) = \cos(x) \qquad f'(\pi/4) = \sqrt{3}/2$$

$$f''(x) = -\sin(x) \qquad f''(\pi/4) = -1/2 f'''(x) \quad = -\cos(x) f'''(\pi/4) \quad = -\sqrt{3}/2$$

So we have

$$T_3(x, \pi/6) = 1/2 + \frac{\sqrt{3}}{2}(x - \pi/6) - \frac{1}{4}(x - \pi/6)^2 - \frac{\sqrt{3}}{12}(x - \pi/6)^3$$

(b) Estimate the error if you use $T_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!}$ to approximate $g(x) = \cos(x)$ at $x = -.5$.

Solution: We can use the remainder theorem. Since $g^{(5)}(x) = -\sin(x)$, we have

$$|R_4(x)| = \left|\frac{f^{(5)}(z)}{5!}x^5\right| = \left|\frac{\sin(z)}{5!}x^5\right| \leq \frac{|x|^5}{5!}$$
$$\frac{.5^5}{120} = \frac{1}{3840} \approx .00026.$$

20. **Power Series as Functions**

(a) Write a power series expression for $\frac{x}{2+x^2}$ centered a 0. What is the radius of convergence?

Solution: We know that

$$\frac{1}{2-x} = \frac{1}{2}\frac{1}{1-x/2} = \frac{1}{2}\sum_{n=0}^{\infty}(x/2)^n$$
$$\frac{1}{2+x^2} = \frac{1}{2}\frac{1}{1-(-x^2/2)} = \frac{1}{2}\sum_{n=0}^{\infty}(-x^2/2)^n = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{2n}$$
$$\frac{x}{2+x^2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{n+1}}x^{2n+1}.$$

The radius of convergence is $\sqrt{2}$. We can figure that out by reasoning from the geometric series: the radius of convergence for the geometric series is 1, so it converges for $-1 < x^2/2 < 1$ or $-2 < x^2 < 2$ or $-\sqrt{2} < x < \sqrt{2}$. Or we can use the ratio test:

$$\lim_{n\to\infty}\left|\frac{x^{2n+3}/2^{n+2}}{x^{2n+1}/2^{n+1}}\right| = \lim_{n\to\infty}\frac{|x|^2}{2}$$

and thus it converges when $x^2/2 < 1$.

(b) If $f(x) = \displaystyle\sum_{n=1}^{\infty}\frac{2^n}{n^2+n}(x-5)^n$, compute $\frac{d}{dx}f(x)$ and $\int f(x)\,dx$.

Solution:

$$\frac{d}{dx}f(x) = \sum_{n=1}^{\infty}\frac{2^n}{n+1}(x-5)^{n-1}$$
$$\text{(or)} \quad = \sum_{n=0}^{\infty}\frac{2^{n+1}}{n}(x-5)^n$$
$$\int f(x)\,dx = \sum_{n=1}^{\infty}\frac{2^n}{(n^2+n)(n+1)}(x-5)^{n+1} + C$$
$$\text{(or)} \quad = \sum_{n=2}^{\infty}\frac{2^{n-1}}{n^2(n-1)}(x-5)^n + C.$$

19. **Power Series**

(a) Find the radius of convergence and the interval of convergence of $\sum_{n=0}^{\infty} \frac{(n!)^2}{(3n)!}(x+2)^n$.

Solution:

We use the ratio test.

$$\lim_{n\to\infty} \left| \frac{((n+1)!)^2(x+2)^{n+1}/(3n+3)!}{(n!)^2(x+2)^n/(3n)!} \right| = \lim_{n\to\infty} |x+2| \frac{(n+1)^2}{(3n+3)(3n+2)(3n+1)} \leq \frac{|x+2|}{n} = 0$$

for any x. So the radius of convergence is infinity, and this converges for all x.

(b) Find the radius of convergence and the interval of convergence of $\sum_{n=1}^{\infty} \frac{3^n(x+2)^n}{\ln(3n)}$.

Solution:

We use the ratio test.

$$\begin{aligned} \lim_{n\to\infty} \left| \frac{3^{n+1}(x+2)^{n+1}/\ln(3n+3)}{3^n(x+2)^n/\ln(3n)} \right| &= \lim_{n\to\infty} 3|x+2| \frac{\ln(3n+3)}{\ln(3n)} \\ &= 3|x+2|. \end{aligned}$$

So we need $3|x+2| < 1$ or $-1 < 3x+6 < 1$, or $-7/3 < x < -5/3$ We need to have x in the interval $(2-1/3, 2+1/3)$, so the radius is $1/3$.

To find the interval we need to check the endpoints. We see

$$\sum_{n=0}^{\infty} \frac{3^n}{\ln(3n)}(1/3)^n \sum_{n=0}^{\infty} \frac{1}{\ln(3n)}$$

converges by comparison to the harmonic series

$$\sum_{n=0}^{\infty} \frac{3^n}{\ln(3n)}(-1/3)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{\ln(3n)}$$

converges by the alternating series test

Thus the interval of convergence is $[-7/3, -5/3)$.

18. **Absolute and Conditional Convergence**

For each series, tell whether it absolutely converges, conditionally converges, or diverges. Justify your answer (and in particular, if it conditionally converges, explain why it doesn't absolutely converge).

(a) $\sum_{n=2}^{\infty} \frac{(-n)^3}{n^4-4}$

Solution: The absolute value series is $\sum \frac{n^3}{n^4-4}$, and $\frac{n^3}{n^4-4} > \frac{1}{n}$. Since the harmonic series $\sum \frac{1}{n}$ diverges, our series does not converge absolutely. But the series itself does converge by the alternating series test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n(3n+1)^n}{(5n^2-2)^n}$

Solution: This series converges by the root test. We take the limit of the nth roots and get

$$\lim_{n\to\infty} \sqrt[n]{\frac{(3n+1)^n}{(5n^2-2)^n}} = \lim_{n\to\infty} \frac{3n+1}{5n^2-2} = 0.$$

This limit is less than 1, so the series converges absolutely.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n+1.1^n}$

Solution: We can use the ratio test here:

$$\lim_{n\to\infty} \left| \frac{{(-1)^{n+1}}/{(n+1+1.1^{n+1})}}{{(-1)^n}/{n+1.1^n}} \right| = \lim_{n\to\infty} \frac{n+1.1^n}{n+1+1.1^{n+1}}$$
$$= \lim_{n\to\infty} \frac{n/1.1^n+1}{n/1.1^n+1/1.1^n+1.1} = \frac{1}{1.1} < 1.$$

So this converges absolutely.

17. **Comparison Tests**

Determine whether each of the following series converges by using an appropriate comparison test.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{n^2}$

Solution:

You can't really use the limit comparison test here, because the obvious comparsion $\sum 1/n^2$ leads to you needing to compute $\lim_{n\to\infty} \sin(1/n)$, which doesn't exist.

But you can use the usual comparison test. We know that

$$\sin(1/n) \leq 1$$
$$\frac{\sin(1/n)}{n^2} \leq \frac{1}{n^2}$$

Since $\sum 1/n^2$ converges, by the comparison test our series also converges.

(In fact it's true that $\sin(1/n) < 1/n$, so $\sum \frac{\sin(1/n)}{n}$ also converges, but that seemed a little hard to see.)

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^3 - n\sin(n)}$

Solution:

Here it would be hard, or at least annoying, to use the regular comparison test. Instead, we limit compare to $\frac{1}{n^2}$. We have

$$\lim_{n\to\infty} \frac{{m}/{n^3 - n\sin(n)}}{{1}/{n^2}} = \lim_{n\to\infty} \frac{n^3}{n^3 - n\sin(n)} = \lim_{n\to\infty} \frac{1}{1-\sin(n)/n^2} = 1.$$

Since this is a finite non-zero number, the two series have the same convergence behavior. Thus, since $\frac{1}{n^2}$ converges, we know that our series also converges.

15. **Geometric and Telescoping Series**

Compute the following infinite sums, with justification:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{5^{n+1}}{3^{2n+1}} =$

Solution:

This is a geometric series with $a = \frac{25}{27}$ and $r = \frac{5}{9}$. Thus the sum is

$$\frac{25/27}{1-5/9} = \frac{25/27}{4/9} = \frac{25}{12}.$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2+11n+30} =$

Solution: This is the same as

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n+5} - \frac{1}{n+6} &= \lim_{t\to\infty} \sum_{n=1}^{t} \frac{1}{n+5} - \frac{1}{n+6} \\ &= \lim_{t\to\infty} \left((1/6 - 1/5) + (1/7 - 1/6) + \dots \right) \\ &= \lim_{t\to\infty} \frac{1}{6} - \frac{1}{t+6} = \frac{1}{6}.\end{aligned}$$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{3n-2}}{7^{n+5}} =$

Solution:

This is a geometric series with $a = \frac{3}{7^6}$ and $r = \frac{27}{7}$. Since $|r| = 27/7 > 1$, this series diverges.

14. **Sequences**

(a) Consider the sequence $(a_n) = (2, 3/4, 4/9, 5/16, 6/25, \dots)$. Find a formula for the nth term a_n. Compute $\lim_{n\to\infty} a_n$.

Solution:

We have $a_n = \frac{n+1}{n^2}$, and thus

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n+1}{n^2} = \lim_{n\to\infty} \frac{1/n + 1/n^2}{1} = 0.$$

(b) Let $b_n = \frac{n!+2}{(n+2)!}$. Compute the first four terms of the sequence, and compute $\lim_{n\to\infty} b_n$, with justification.

Solution:

$$b_1 = \frac{3}{6} = \frac{1}{2} \qquad b_2 = \frac{4}{24} = \frac{1}{6}$$
$$b_3 = \frac{8}{120} = \frac{1}{15} \qquad b_4 = \frac{26}{720} = \frac{13}{360}.$$

To compute the limit, we have

$$\begin{aligned}
\lim_{n\to\infty} b_n &= \lim_{n\to\infty} \frac{n!+2}{(n+2)!} \\
&= \lim_{n\to\infty} \frac{1+2/n!}{(n+2)(n+1)} = 0 \\
\text{(or better)} \quad &= \lim_{n\to\infty} \frac{\frac{1}{(n+1)(n+1)} + \frac{2}{(n+2)!}}{1} = \frac{0}{1} = 0.
\end{aligned}$$

(c) Let $c_n = \frac{\sqrt{n}}{\sqrt{n+2}}$. Compute the first four terms of this sequence, and compute $\lim_{n\to\infty} c_n$, with justification.

Solution:

$$c_1 = \frac{1}{\sqrt{3}} \qquad c_2 = \sqrt{2}/2$$

$$c_3 = \sqrt{3}/\sqrt{5} \qquad c_4 = 2/\sqrt{6}$$

We can copute:

$$\lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n+2}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+2/n}} = 1.$$

12. **Arc Length and Surface Area**

(a) Compute the arc length of the curve $y = \frac{1}{27}(9x^2+6)^{3/2}$ as x varies from 2 to 4.

Solution: We have $y' = x\sqrt{9x^2+6}$, and thus

$$\begin{aligned}
L &= \int_2^4 \sqrt{1+x^2(9x^2+6)}\,dx = \int_2^4 \sqrt{1+6x^2+9x^4}\,dx \\
&= \int_2^4 3x^2+1 = x^3+x\Big|_2^4 = 64+4-8-2 = 58
\end{aligned}$$

(b) Compute the area of the surface obtained by taking the curve $y = x^3$ as x goes from 0 to 1 and rotating it around the x-axis.

Solution: We have $y' = 3x^2$, and so

$$\begin{aligned}
L &= \int_0^1 2\pi x^3 \sqrt{1+(3x^2)^2}\,dx = \int_0^1 2\pi x^3\sqrt{1+9x^4}\,dx \\
&= \frac{\pi}{27}(1+9x^4)^{3/2}\Big|_0^1 = \frac{10^{3/2}\pi}{27} - \frac{\pi}{27}.
\end{aligned}$$

11. **Improper Integrals**

(a) Compute $\displaystyle\int_1^\infty \frac{\ln(x)}{x}\,dx$.

Solution: We have

$$\int_1^\infty \frac{\ln(x)}{x}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln(x)}{x}\,dx \qquad = \lim_{t\to\infty}\frac{1}{2}\ln(x)^2\Big|_1^t$$
$$= \lim_{t\to\infty}\frac{1}{2}\ln(t)^2 - 0 = \infty.$$

Thus the integral diverges.

(b) Compute $\int_{-1}^1 \frac{1}{\sqrt[3]{x^2}}\,dx$.

Solution:

$$\begin{aligned}\int_{-1}^1 \frac{1}{\sqrt[3]{x}}\,dx &= \lim_{t\to 0^-}\int_{-1}^t \frac{1}{\sqrt[3]{x^2}}\,dx + \lim_{s\to 0^+}\int_s^1 \frac{1}{\sqrt[3]{x^2}}\,dx \\ &= \lim_{t\to 0^-} 3x^{1/3}\Big|_{-1}^t + \lim_{s\to 0^+} 3x^{1/3}\Big|_s^1 \\ &= \lim_{t\to 0^-} 3t^{1/3} - 3(-1)^{1/3} + \lim_{s\to 0^+} 3\cdot 1^{1/3} - 3s^{1/3} \\ &= 0+3+3-0 = 6.\end{aligned}$$

10. **Numeric Integration**

(a) How many intervals do you need with the **trapezoid** rule to approximate $\int_0^3 \frac{1}{1+x}$ to within 1/2? Compute that approximation. (Feel free to use a calculator to plug in numeric values, or to leave the answer in exact unsimplified terms, but show every step.)

Solution:

We have

$$\begin{aligned} f'(x) &= \frac{-1}{(1+x)^2} \\ f''(x) &= \frac{2}{(x+1)^3} \\ |f''(x)| &\le \frac{2}{1^3} = 2 \end{aligned}$$

Then we have

$$|E_T| \le \frac{2(3-0)^3}{12n^2} = \frac{9}{2n^2}$$

We want $1/2 \ge \frac{9}{2n^2}$ which implies

$$\begin{aligned} n^2 &\ge 9 \\ n &\ge 3. \end{aligned}$$

Then we have

$$T_3 = \frac{1}{2}f(0) + f(1) + f(2) + \frac{1}{2}f(3) = \frac{1}{2} + \frac{1}{2} + \frac{1}{3} + \frac{1}{8} = \frac{35}{24} \approx 1.45833$$

Since the true answer is $\ln(4) \approx 1.38629$, this is indeed within our margin of error.

(b) Suppose we have

$$g(0) = 2 \quad g(.5) = 3 \quad g(1) = 4 \quad g(1.5) = 5 \quad g(2) = 3 \quad g(2.5) = 2 \quad g(3) = 1$$

Approximate $\int_0^3 g(x)\,dx$ using the midpoint rule and Simpson's rule.

Solution:

For the midpoint rule, we have

$$M_3 = g(.5) + g(1.5) + g(2.5) = 3 + 5 + 2 = 10$$

For the Simpson's rule, we have

$$S_6 = \frac{1}{6}(2 + 12 + 8 + 20 + 6 + 8 + 1) = \frac{57}{6} = \frac{19}{2} = 9.5.$$

4. **Integrals Involving Exponentials and Logarithms**

Compute the following integrals:

(a) $\int \sec^2(x)e^{\tan(x)}\,dx =$

Solution: We take $u = \tan(x)$ so $du = \sec^2(x)\,dx$. Then

$$\int \sec^2(x)e^{\tan(x)}\,dx = \int e^u\,du \qquad = e^u + C = e^{\tan(x)} + C.$$

(b) $\int \frac{1}{x\ln(x)\ln(\ln(x))}\,dx =$

Solution:

Take $u = \ln(\ln(x))$ so $du = \frac{dx}{x\ln(x)}$. Then

$$\int \frac{1}{x\ln(x)\ln(\ln(x))}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln(\ln(ln(x))) + C.$$

(c) $\int_0^2 x2^{x^2}\,dx$

Solution:

We can take $u = x^2$ so that $du = 2x\,dx$. Then

$$\int_0^2 x2^{x^2}\,dx = \int_0^4 \frac{1}{2}2^u\,du \qquad = \frac{1}{2}2^u\ln(2)\Big|_0^8 = \frac{1}{2}(2^4\ln(2) - \ln(2)) = \frac{15\ln(2)}{2}.$$

1. **Topic 1: Inverse Functions**

(a) Is $f(x) = \sqrt{x^4 + 3}$ invertible or not? Justify your answer.

Solution: We have $f(-1) = f(1) = 2$ so thsi function is not one-to-one, and thus not invertible.

(b) Find a formula for the inverse of $g(x) = \ln(x^3 - 2)$.

Solution:

$$\begin{aligned} y &= \ln(x^3 - 2) \\ e^y &= x^3 - 2 \\ e^y + 2 &= x^3 \\ x &= \sqrt[3]{e^y + 2}. \end{aligned}$$

so $g^{-1}(y) = \sqrt[3]{3^y + 2}$. (You can use whichever variable you like in your formula.)

(c) Let $h(x) = \sqrt{5x^3 + 3x + 1}$. Compute $(h^{-1})'(3)$.

Solution: By the Inverse Function Theorem, we know that

$$(h^{-1})'(3) = \frac{1}{h'(h^{-1}(3))}.$$

Guess and check shows that $h(1) = 3$ so $h^{-1}(3) = 1$. And we know that

$$h'(x) = \frac{1}{2}(5x^3 + 3x + 1)^{-1/2}(15x^2 + 3)$$

and thus

$$h'(2) = \frac{1}{6}(18) = 3.$$

Thus

$$(h^{-1})'(3) = \frac{1}{3}.$$