Math 1232 Sprin 2021
Single-Variable Calculus II Mastery Quiz 3
Due Friday, February 5

This week's mastery quiz has six topics. You should do topics 6 and 5, and optionally one of the previous topics. Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 20-30 minutes on this quiz.

Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

6. L'Hospital's Rule
5. Inverse Trigonometric Functions
4. Integrals involving Exponentials and Logarithms
3. Derivatives of Exponentials and Logarithms
2. The Exponential and the Logarithm
1. Inverse Functions

6. **L'Hospital's Rule**

Compute the following limits:

(a) $\lim_{x\to 2} \dfrac{e^{x^2-4} - x + 1}{x-2} =$

Solution:

The limit of the top and bottom are both 0, we can use L'Hospital's rule.

$$\lim_{x\to 2} \frac{e^{x^2-4} - x + 1}{x-2} = \lim_{x\to 2} \frac{2xe^{x^2-4} - 1}{1} = 3.$$

(b) $\lim_{x\to\infty} = x^{\frac{2}{1+\ln(x)}}$

Solution:

$$\ln y = \frac{2}{1+\ln(x)} \ln(x)$$
$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{2\ln(x)}{1+\ln(x)}$$

The top and bottom both go to ∞, so we can use L'Hospital's Rule:

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{2/x}{1/x} = 2$$

and thus

$$\lim_{x\to\infty} y = e^2.$$

(c) $\lim_{x\to 0} \dfrac{x^3 - x^2}{x + \sin(x)} =$

Solution: The limits of the top and bottom are both zero, so we can use L'Hospital's Rule:

$$\lim_{x\to 0} \frac{x^3 - x^2}{x + \sin(x)} = \lim_{x\to 0} \frac{3x^2 - 2x}{1 + \cos(x)} = \frac{0}{2} = 0.$$

Note we *cannot* use L'Hospital's rule a second time, because we don't have an indeterminate form.

5. **Inverse Trigonometric Functions**

(a) Compute $\arccos\left(\frac{-\sqrt{3}}{2}\right) =$

Solution: $5\pi/6$.

(b) Compute $\tan(\arcsin(.4))$.

Solution:

If we draw a triangle, we have opposite side with length 4 and hypotenuse with length 10. By the Pythagorean theorem, the adjacent side will have length $\sqrt{86}$, and thus $\tan(\theta) = \frac{4}{\sqrt{86}}$.

(c) $\frac{d}{dx}\arccos(x+e^x) =$

Solution:

$$\frac{d}{dx}\arcsin(x+e^x) = \frac{1+e^x}{\sqrt{1-(x+e^x)^2}}.$$

(d) $\int \frac{x}{9+x^4}\,dx =$

Solution:

We can factor a 9 out to get $\frac{1}{9}\frac{x}{1+x^4/9}$. Then we set $u = x^2/3$, and $du = 2x/3\,dx$, and we have

$$\begin{aligned}\int \frac{x}{9+x^4}\,dx &= \int \frac{1}{9}\frac{x}{1+u^2}\frac{3}{2x}\,du \\ &= \int \frac{1}{6}\frac{1}{1+u^2}\,du \\ &= \frac{1}{6}\arctan u + C = \frac{1}{6}\arctan(x^2/3) + C.\end{aligned}$$

4. **Integrals Involving Exponentials and Logarithms**

Compute the following integrals:

(a) $\int \frac{1}{s\log_3(s)}\,ds =$

Solution:

Set $u = \log_3(s)$ so $du = \frac{1}{s\ln(3)}\,ds$, and we have

$$\int \frac{1}{s\log_3(s)}\,ds = \int \frac{\ln(3)du}{u} = \ln(3)\ln|u| + C = \ln(3)\ln|\log_3(s)| + C.$$

(b) $\int e^x\sqrt{1-3e^x}\,dx$

Solution:

Set $u = 1-3e^x$, so $du = -3e^x\,dx$, and we have

$$\begin{aligned}\int e^x\sqrt{1-3e^x}\,dx &= \int \frac{-1}{3}\sqrt{u}\,du \\ &= \frac{-2}{9}u^{3/2} + C = \frac{-2}{9}(1-3e^x)^{3/2} + C.\end{aligned}$$

(c) $\int \cot(5t)\,dt =$

Solution:

This is a lot like the integral of tangent. We c an think of $\cot(5t)$ as $\cos(5t)/\sin(5t)$. If we take $u = \sin(5t)$ then $du = 5\cos(5t)\,dt$, and we get

$$\begin{aligned}\int \cot(5t)\,dt &= \int \frac{\cos(5t)}{\sin(5t)}\,dt = \int \frac{1}{5}\frac{1}{u}\,du \\ &= \frac{1}{5}\ln|u| + C = \frac{1}{5}\ln|\sin(5t)| + C.\end{aligned}$$

3. **Derivatives of Exponentials and Logarithms**

   (a) Compute $\frac{d}{dx}x^{e^x}$

   Solution:

$$\begin{aligned} y &= x^{e^x} \\ ln|y| &= e^x \ln|x| \\ y'/y &= e^x \ln|x| + \frac{e^x}{x} \\ y' &= e^x \ln|x| x^{e^x} + \frac{1}{x} e^x x^{e^x}. \end{aligned}$$

   (b) Find an equation for the tangent line to the curve $y = 3^{x^2+1}$ at the point $(1, 9)$.

   Solution:

   We have $y' = 3^{x^2+1}\ln(3) \cdot 2x$ and thus $y'(1) = 9\ln(3) \cdot 2$. So the equation of the tangent line is

$$y - 9 = 18\ln(3)(x - 1).$$

   (c) Compute $\frac{d}{du}\ln(4^u + \sqrt{u})$.

   Solution:

$$\frac{d}{du}\ln(4^u + \sqrt{u}) = \frac{1}{4^u + \sqrt{u}}\left((4^u \ln(4) + \frac{1}{2\sqrt{u}}\right)$$

2. **Topic 2: Exponents and Logarithms**

   (a) Showing your work, compute $\log_8(12) - \log_8(15) + \log_3(20)$. (Give an exact answer with no decimals.)

   Solution:

$$\log_8(12) - \log_8(15) + \log_3(20) = \log_8(16) = 4/3.$$

   (b) Give an exact solution for the equation $4^{x^3+1} = 5$.

   Solution:

$$\begin{aligned} x^3 + 1 &= \log_4(5) \\ x^3 &= \log_4(5) - 1 \\ x &= \sqrt[3]{\log_4(5) - 1}. \end{aligned}$$

   (c) Compute $3^{\log_3(15) - 2log_3(6)}$. (Give an exact answer with no decimals.)

   Solution:

$$\begin{aligned} 3^{\log_3(15) - 2log_3(6)} &= \frac{3^{\log_3(15)}}{3^{2\log_3(6)}} \\ &= \frac{15}{6^2} = \frac{5}{12}. \end{aligned}$$

(d) Give an exact solution for the equation $\ln(3 - 5x) = 2$.
Solution:

$$\begin{aligned} 3 - 5x &= e^2 \\ 5x &= 3 - e^2 \\ x &= \frac{3 - e^2}{5}. \end{aligned}$$

1. **Topic 1: Inverse Functions**

(a) Is $f(x) = e^{x^2}$ invertible or not? Justify your answer.
Solution: We have $f(-1) = f(1) = e$ so thsi function is not one-to-one, and thus not invertible.

(b) Find a formula for the inverse of $g(x) = (x^3 + 3)^3$.
Solution:

$$\begin{aligned} y &= (x^3 + 3)^3 + \sqrt[3]{y} && = x^3 + 3 \\ \sqrt[3]{y} - 3 &= x^3 \\ x &= \sqrt[3]{\sqrt[3]{y} - 3} \end{aligned}$$

so $g^{-1}(y) = \sqrt[3]{\sqrt[3]{y} - 3}$. (You can use whichever variable you like in your formula.)

(c) Let $h(x) = \sqrt{x^5 + x + 2}$. Compute $(h^{-1})'(6)$.
Solution: By the Inverse Function Theorem, we know that

$$(h^{-1})'(2) = \frac{1}{h'(h^{-1}(6))}.$$

Guess and check shows that $h(2) = 6$ so $h^{-1}(6) = 2$. And we know that

$$h'(x) = \frac{1}{2}(x^5 + x + 2)^{-1/2}(5x^4 + 1)$$

and thus

$$h'(2) = \frac{1}{12}(81)$$

Thus

$$(h^{-1})'(6) = \frac{12}{81}.$$