Math 1232 Sprin 2021
Single-Variable Calculus II Mastery Quiz 4
Due Friday, February 12

This week's mastery quiz has eigh topics. You should do topics 8 and 7, and optionally one of the previous topics. Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 20-30 minutes on this quiz.

Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

8. Trigonometric integrals
7. Integration by parts
6. L'Hospital's Rule
5. Inverse Trigonometric Functions
4. Integrals involving Exponentials and Logarithms
3. Derivatives of Exponentials and Logarithms
2. The Exponential and the Logarithm
1. Inverse Functions

8. **Trigonometric Integrals**

   Compute

   (a) $\int \cos^3(2x)\,dx =$

   Solution:

$$\begin{aligned}\int \cos^3(2x)\,dx &= \int \cos(2x)(1-\sin^2(2x))\,dx\\ &= \int \cos(2x) - \sin^2(2x)\cos(2x)\,dx\\ &\frac{1}{2}\sin(2x) - \frac{1}{6}\sin^3(2x) + C.\end{aligned}$$

   (b) $\int \frac{\sqrt{4x^2-1}}{x}\,dx =$

   Solution: We set $2x = \sec(\theta)$, so $dx = \frac{1}{2}\sec(\theta)\tan(\theta)\,d\theta$, and

$$\begin{aligned}\int \frac{\sqrt{4x^2-1}}{x}\,dx &= \int \frac{\sqrt{\sec^2\theta - 1}}{\frac{1}{2}\sec\theta}\frac{1}{2}\sec(\theta)\tan\theta\,d\theta\\ &= \int \tan^2(\theta)\,d\theta = \int \sec^2(\theta) - 1\,d\theta\\ &= \tan(\theta) - \theta + C\end{aligned}$$

   Then we know $\sec\theta = 2x$ so we can make a triangle with hypotenuse $2x$ and adjacent side 1, and thus opposite side $\sqrt{4x^2-1}$, so $\tan(\theta) = \sqrt{4x^2-1}$. Then we can say either $\theta = \text{arcsec}(2x)$ or $\theta = \arctan(\sqrt{4x^2-1})$, and we have

$$\int \frac{\sqrt{4x^2-1}}{x}\,dx = \sqrt{4x^2-1} - \arctan(\sqrt{4x^2-1}) + C = \sqrt{4x^2-1} - \text{arcsec}(2x) + C.$$

7. **Integration by Parts**

   Compute:

   (a) $\int_0^3 x^2 e^{2x}\,dx =$

Solution:

$$\begin{aligned}
\int_0^3 x^2 e^{2x}\,dx &= \frac{1}{2}x^2 e^{2x}\Big|_0^3 - \int_0^3 x e^{2x}\,dx \\
&= \frac{9}{2}e^6 - \int_0^3 x e^{2x}\,dx \\
\int_0^3 x e^{2x}\,dx &= \frac{1}{2}x e^{2x}\Big|_0^3 - \int_0^3 \frac{1}{2}e^{2x}\,dx \\
&= \frac{1}{2}x e^{2x} - \frac{1}{4}e^{2x}\Big|_0^3 \\
&= \frac{3}{2}e^6 - \frac{1}{4}e^6 + \frac{1}{4} \\
\int_0^3 x^2 e^{2x}\,dx &= \frac{9}{2}e^6 - \frac{3}{2}e^6 + \frac{1}{4}e^6 - \frac{1}{4} = \frac{13}{4}e^6 - \frac{1}{4}.
\end{aligned}$$

(b) $\int \sin(2x)\cos(3x)\,dx =$

Solution:

$$\begin{aligned}
\int \sin(2x)\cos(3x)\,dx &= \frac{1}{3}\sin(2x)\sin(3x) - \int \frac{2}{3}\cos(2x)\sin(3x)\,dx \\
\int \cos(2x)\sin(3x)\,dx &= -\frac{1}{3}\cos(2x)\cos(3x) - \int \frac{2}{3}\sin(2x)\cos(3x) \\
\int \sin(2x)\cos(3x)\,dx &= \frac{1}{3}\sin(2x)\sin(3x) + \frac{2}{9}\cos(2x)\cos(3x) + \frac{4}{9}\int \sin(2x)\cos(3x)\,dx \\
\frac{5}{9}\int \sin(2x)\cos(3x)\,dx &= \frac{1}{3}\sin(2x)\sin(3x) + \frac{2}{9}\cos(2x)\cos(3x)\big(+C\big) \\
\int \sin(2x)\cos(3x)\,dx &= \frac{3}{5}\sin(2x)\sin(3x) + \frac{2}{5}\cos(2x)\cos(3x) + C.
\end{aligned}$$

6. **L'Hospital's Rule**

Compute the following limits:

(a) $\lim_{x\to -1} \frac{x^2-2x-3}{x^2+5x+4} =$

Solution:

The top and the bottom both go to 0, so we can us L'Hospital's Rule

$$\lim_{x\to -1} \frac{x^2-2x-3}{x^2+5x+4} = \lim_{x\to -1} \frac{2x-2}{2x+5} = \frac{-4}{3}.$$

(b) $\lim_{x\to 0} \left(\frac{e^x+1}{2}\right)^{1/x} =$

Solution:

$$
\begin{aligned}
\ln y &= \frac{1}{x}\ln\left(\frac{e^x+1}{2}\right) \\
\lim_{x\to 0}\ln y &= \lim_{x\to 0}\frac{\ln\left(\frac{e^x+1}{2}\right)}{x}
\end{aligned}
$$

Here the top and the bottom both go to 0, so we can use L'Hospital's Rule:

$$
\begin{aligned}
\lim_{x\to 0}\ln y &= \lim_{x\to 0}\frac{2}{e^x+1}\cdot e^x/2 = \lim_{x\to 0}\frac{e^x}{e^x+1} = 1/2 \\
\lim_{x\to 0} y &= e^{1/2}.
\end{aligned}
$$

(c) $\lim_{x\to 1}\frac{\ln(x)}{\arcsin(x-1)} =$

Solution:

The top and bottom both approahc 0, so we can use L'Hospital's Rule:

$$
\begin{aligned}
\lim_{x\to 1}\frac{\ln(x)}{\arcsin(x-1)} &= \lim_{x\to 1}\frac{1/x}{\frac{1}{\sqrt{1-(x-1)^2}}} \\
&= \lim_{x\to 1}\frac{\sqrt{1-(x-1)^2}}{x} = 1.
\end{aligned}
$$

5. **Inverse Trigonometric Functions**

(a) Compute $\arctan\left(\sqrt{3}\right) =$

Solution: $\pi/3$.

(b) Compute $\sin(\arccos(5/13))$.

Solution:

If we draw a triangle, we have adjacent side with length 5 and hypotenuse with length 13. By the Pythagorean theorem, the opposite side will have length $\sqrt{144} = 12$, and thus $\sin(\theta) = \frac{12}{13}$.

(c) $\frac{d}{dx}\left(\arctan(x^2)\right)^2 =$

Solution:

$$
\frac{d}{dx}\left(\arctan(x^2)\right)^2 = 2\arctan(x^2)\frac{2x}{1+x^4}.
$$

(d) $\int \frac{1}{\sqrt{4-x^2}}\,dx =$

Solution:

We can factor a 4 out to get $\frac{1}{2}\frac{1}{\sqrt{1-x^2/4}}$. Then we set $u = x/2$, and $du = 1/2\,dx$, and we have

$$
\begin{aligned}
\int \frac{1}{\sqrt{4-x^2}}\,dx &= \int \frac{1}{2}\frac{2}{\sqrt{1-u^2}}\,du \\
&= \int \frac{1}{\sqrt{1-u^2}}\,du \\
&= \arcsin u + C = \arcsin(x/2) + C.
\end{aligned}
$$

4. **Integrals Involving Exponentials and Logarithms**

Compute the following integrals:

(a) $\displaystyle\int \frac{x+3}{x+1}\,dx =$

Solution:

$$\int \frac{x+3}{x+1}\,dx = \int 1 + \frac{2}{x+1}\,dx = x +_2 \ln|x+1| + C.$$

(b) (Note this is a definite integral)

$\displaystyle\int_0^2 \frac{e^x}{e^x+1}\,dx =$

Solution: We can take $u = e^x$ so $du = e^x\,dx$ and

$$\int_0^2 \frac{e^x}{e^x+1}\,dx = \int_1^{e^2} \frac{1}{u+1}\,du = \ln|u+1|\Big|_1^{e^2} = \ln(e^2+1) - \ln(2).$$

(c) $\displaystyle\int 3^{3x+1}\,dx =$

Solution:

We can take $u = 3x+1$ so $du = 3\,dx$, and

$$\int 3^{3x+1}\,dx = \int 3^u \frac{1}{3}\,du = \frac{3^u}{\ln(3)}\frac{1}{3} + C = \frac{3^{3x+1}}{3\ln(3)} + C\Big(= \frac{1}{\ln(3)}3^{3x} + C\Big).$$

3. **Derivatives of Exponentials and Logarithms**

(a) Compute $\displaystyle\frac{d}{dx}\sqrt{x+1}^x$

Solution:

$$\begin{aligned} y &= \sqrt{x+1}^x \\ ln|y| &= x\ln(\sqrt{x+1}) = \frac{1}{2}x\ln(x+1) \\ y'/y &= \frac{1}{2}\left(\ln(x+1) + \frac{x}{x+1}\right) \\ y' &= \frac{1}{2}\sqrt{x+1}^x\left(\ln(x+1) + \frac{x}{x+1}\right) \end{aligned}$$

(b) Find an equation for the tangent line to the curve $y = \log_5(x^2+1)$ at the point $(2,1)$.

Solution:

We have $y' = \frac{2x}{(x^2+1)\ln(5)}$ and thus $y'(2) = \frac{4}{5\ln(5)}$. So the equation of the tangent line is

$$y - 1 = \frac{4}{5\ln(5)}(x-2).$$

(c) Compute $\frac{d}{dz}4^{z^2}\ln(z)$.
Solution:

$$\frac{d}{dz}4^{z^2}\ln(z) = 4^{z^2}2z\ln(4)\ln(z) + \frac{1}{z}4^{z^2}.$$

2. **Topic 2: Exponents and Logarithms**

(a) Showing your work, compute $2\log_{1/2}(6) - \log_{1/2}(63) + \log_{1/2}(14)$. (Give an exact answer with no decimals.)
Solution:

$$2\log_{1/2}(6) - \log_{1/2}(21) + \log_{1/2}(14) = \log_{1/2}\left(6^2 cdot 14/63\right) = \log_{1/2}(8) = -3.$$

(b) Compute $5^{3\log_5(10)-2log_5(7)}$. (Give an exact answer with no decimals.)
Solution:

$$\begin{aligned} 5^{3\log_5(10)-2log_5(7)} &= \frac{5^{3\log_5(10)}}{5^{2\log_5(7)}} \\ &= \frac{10^3}{7^2}. \end{aligned}$$

(c) Give an exact solution for the equation $e^{2x-5} = 4$.
Solution:

$$\begin{aligned} 2x - 5 &= \ln(4) \\ 2x &= \ln(4) + 5 \\ x &= \frac{1}{2}(\ln(4) + 5) \end{aligned}$$

(d) Give an exact solution for the equation $\log_{31}(x^2 - 7) = 3$.
Solution:

$$\begin{aligned} x^2 - 7 &= 31^3 \\ x^2 &= 31^3 + 7 \\ x &= \pm\left(31^3 + 7\right) \end{aligned}$$

1. **Topic 1: Inverse Functions**

(a) Is $f(x) = \sqrt{x^4+1}$ invertible or not? Justify your answer.
Solution: We have $f(-1) = f(1) = \sqrt{2}$ so thsi function is not one-to-one, and thus not invertible.

(b) Find a formula for the inverse of $g(x) = \sqrt{x^5 - 7}$.

Solution:

$$\begin{aligned} y &= \sqrt{x^5 - 7} \\ y^2 &= x^5 - 7 \\ y^2 + 7 &= x^5 \\ x &= \sqrt[5]{y^2 + 7} \end{aligned}$$

so $g^{-1}(y) = \sqrt[5]{y^2 + 7}$.

(c) Let $h(x) = x^7 + 3x^3 + 1$. Compute $(h^{-1})'(5)$.

Solution: By the Inverse Function Theorem, we know that

$$(h^{-1})'(5) = \frac{1}{h'(h^{-1}(5))}.$$

Guess and check shows that $h(1) = 5$ so $h^{-1}(5) = 1$. And we know that

$$h'(x) = 7x^6 + 9x^2$$

and thus

$$h'(1) = 7 + 9 = 16$$

Thus

$$(h^{-1})'(5) = \frac{1}{16}.$$