Math 1232 Sprin 2021
Single-Variable Calculus II Mastery Quiz 6
Due Friday, February 26

This week's mastery quiz has seven topics. You should do topics 12 and 11, and optionally one of the previous topics. Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 20-30 minutes on this quiz.

Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

12. Arc Length and Surface Area
11. Improper Integrals
10. Numeric Integration
9. Partial Fractions
8. Trigonometric integrals
7. Integration by parts
4. Integrals involving Exponentials and Logarithms

12. **Arc Length and Surface Area**

(a) Compute the arc length of the curve $(y-2)^3 = x^2$ between $y = 2$ and $y = 6$.

Solution:

We have $x = (y-2)^{3/2}$, so $\frac{dx}{dy} = \frac{3}{2}(y-2)^{1/2}$ and

$$\begin{aligned} L &= \int_2^6 \sqrt{1 + \frac{9}{4}(y-2)}\, dy \\ &= \frac{8}{27}\left(1 + \frac{9}{4}(y-2)\right)^{3/2}\Big|_2^6 \\ &= \frac{8}{27}\left(10^{3/2} - 1\right). \end{aligned}$$

(b) Set up (but don't compute!) an integral for the area of a surface obtained by taking the curve $y = \ln(x^3+1)$ from $x = 0$ to $x = 10$ and rotating around the x axis.

Solution: We have $y' = \frac{3x^2}{x^3+1}$ so we get

$$\begin{aligned} SA &= \int_0^{10} 2\pi y \sqrt{1 + y'^2}\, dx \\ &= \int_0^{10} 2\pi \ln(x^3+1) \sqrt{1 + \frac{9x^4}{(x^3+1)^2}}\, dx. \end{aligned}$$

11. **Improper Integrals**

(a) Compute $\displaystyle\int_1^\infty \frac{\ln(x)}{x^2}\, dx =$

Solution:

$$\begin{aligned} \int_1^\infty \frac{\ln(x)}{x^2}\, dx &= \lim_{t\to\infty} \int_1^t \frac{\ln(x)}{x^2}\, dx \\ &= \lim_{t\to\infty} \frac{-\ln(x)}{x}\Big|_1^t - \int_1^t \frac{-1}{x^2}\, dx \\ &= \lim_{t\to\infty} \frac{-\ln(x)}{x}\Big|_1^t - \frac{1}{x}\Big|_1^t \\ &\lim_{t\to\infty} \frac{-\ln(t)}{t} + 0 - \frac{1}{t} + 1 \\ &= 1 - \lim_{t\to\infty} \frac{\ln(t)}{t} \\ &= 1 - \lim_{t\to\infty} \frac{1/t}{1} = 1. \end{aligned}$$

(b) Compute $\displaystyle\int_1^3 \frac{x}{x^2-1}\, dx =$

Solution:

$$\begin{aligned}\int_1^3 \frac{x}{x^2-1}\,dx &= \lim_{t\to1^+}\int_t^3 \frac{x}{x^2-1}\,dx \\ &= \lim_{t\to1^+}\frac{1}{2}\ln|x^2-1|\Big|_t^3 \\ &= \lim_{t\to1^+}\ln(8)-\ln|t^2-1| \\ &= \ln(8)-\lim_{x\to1^+}\ln|t^2-1| = \infty.\end{aligned}$$

So this integral doesn't converge.

10. **Numeric Integration**

(a) How many intervals do you need with the **trapezoid** rule to approximate $\int_5^9 (x+4)^{3/2}\,dx$ to within 1/10? Compute that approximation. (Feel free to use a calculator to plug in numeric values, or to leave the answer in exact unsimplified terms, but show every step.)

Solution:

We have

$$\begin{aligned} f'(x)\frac{3}{2}(x+4)^{1/2} f''(x) &= \frac{3}{4}(x+4)^{-1/2} = \frac{3}{4\sqrt{x+4}} \\ f''(5) &= \frac{1}{4} \\ |E_M| &\leq \frac{1/4\cdot 4^3}{12\cdot n^2} \leq \frac{1}{10} \\ n^2 &\geq 40/3 \approx 13.3 \\ n &\geq 4 \end{aligned}$$

so we need to use at least four intervals. Then the midpoint approximation would be

$$\begin{aligned}\int_5^9 (x+4)^{3/2}\,dx &\approx \frac{\sqrt{9}^3+\sqrt{10}^3}{2} + \frac{\sqrt{10}^3+\sqrt{11}^3}{2} + \frac{\sqrt{11}^3+\sqrt{12}^3}{2} + \frac{\sqrt{12}^3+\sqrt{13}^3}{2} \\ &\approx \frac{1}{2}9^{3/2} + 10^{3/2} + 11^{3/2} + 12^{3/2} + \frac{1}{2}13^{3/2}.\end{aligned}$$

We can stop there, but numerically this is roughly 146.61. The true answer is approximately 146.54 so this is within the expected error bound.

(b) Suppose we have

$$g(0)=5 \quad g(1)=4 \quad g(2)=7 \quad g(3)=4 \quad g(4)=2 \quad g(5)=3 \quad g(6)=5$$

Approximate $\int_0^4 g(x)\,dx$ using the Midpoint rule and using Simpson's rule.

Solution:

I had *meant* to ask for the integral from 0 to 6. If I'd successfully written that, the solution would have been:

For the midpoint rule, we have

$$M_3 = 2 \cdot g(1) + 2 \cdot g(3) + 2 \cdot g(5) = 2 \cdot 4 + 2 \cdot 4 + 2 \cdot 3 = 22.$$

For Simpson's rule, we have

$$\begin{aligned} S_6 &= \frac{1}{3}\left(5 + 4 \cdot 4 + 2 \cdot 7 + 4 \cdot 4 + 2 \cdot 2 + 4 \cdot 3 + 5\right) \\ &= \frac{1}{3}(5 + 16 + 14 + 16 + 4 + 12 + 5) = \frac{1}{3} \cdot 72 = 24 \end{aligned}$$

For the problem I actually wrote, we instead have:
For the midpoint rule, we have

$$M_2 = 2 \cdot g(1) + 2 \cdot g(3) = 2 \cdot 4 + 2 \cdot 4 = 16.$$

For Simpson's rule, we have

$$\begin{aligned} S_4 &= \frac{1}{3}\left(5 + 4 \cdot 4 + 2 \cdot 7 + 4 \cdot 4 + 1 \cdot 2\right) \\ &= \frac{1}{3}(5 + 16 + 14 + 16 + 2) = \frac{1}{3} \cdot 53 = {}^{53}\!/_{2} = 26.5. \end{aligned}$$

9. **Partial Fractions**

(a) Compute $\int \frac{x^2+x+3}{x-2}\,dx =$
Solution: Polynomial long division gives that

$$\begin{aligned} x^2 + x + 3 &= x(x-2) + 3(x-2) + 9 \\ \frac{x^2+x+3}{x-2} &= x + 3 + \frac{9}{x-2} \\ \int \frac{x^2+x+3}{x-2}\,dx &= \int x + 3 + \frac{9}{x-2}\,dx \\ &= \frac{x^2}{2} + 3x + 9\ln|x-2| + C. \end{aligned}$$

(b) Compute $\int \frac{x^2+x-4}{(x+3)^2(x+1)}\,dx =$
Solution:

$$\begin{aligned} \frac{x^2+x-4}{(x+3)^2(x+1)} &= \frac{A}{x+3} + \frac{B}{(x+3)^2} + \frac{C}{x+1} \\ x^2 + x - 4 &= A(x+3)(x+1) + B(x+1) + C(x+3)^2 \\ 2 &= -2B \Rightarrow B = -1 \\ -4 &= 4C \Rightarrow C = -1 \\ -4 &= 3A + B + 9C = 3A - 1 - 9 \Rightarrow A = 2 \\ \frac{x^2+x-4}{(x+3)^2(x+1)} &= \frac{2}{x+3} + \frac{-1}{(x+3)^2} + \frac{-1}{x+1} \\ \int \frac{x^2+x-4}{(x+3)^2(x+1)}\,dx &= \int \frac{2}{x+3} + \frac{-1}{(x+3)^2} + \frac{-1}{x+1}\,dx \\ &= 2\ln|x+3| + \frac{1}{x+3} - \ln|x+1| + C. \end{aligned}$$

8. **Trigonometric Integrals**

Compute

(a) $\displaystyle\int_0^{\pi/6} \sec^3(2t)\tan(2t)\,dt =$

Solution:

We're going to take $u = \sec(2t)$ so that $du = 2\sec(2t)\tan(2t)\,dt$. We compute $u(0) = 1$ and $u(\pi/6) = \sec(\pi/3) = 2$. Then

$$\begin{aligned}\int_0^{\pi/6} \sec^3(2t)\tan(2t)\,dt &= \int_1^2 \frac{1}{2}u^2\,du \\ &= \frac{u^3}{6}\Big|_1^2 = \frac{8}{6} - \frac{1}{6} = \frac{7}{6}.\end{aligned}$$

(b) $\displaystyle\int (x-1)^3\sqrt{2x-x^2}\,dx =$

Solution:

We're going to set $x - 1 = \sin\theta$ so that $dx = \cos\theta\,d\theta$. Then

$$\begin{aligned}\int (x-1)^3\sqrt{2x-x^2}\,dx \int (x-1)^3\sqrt{1-(x-1)^2}\,dx \\ &= \int \sin^3(\theta)\sqrt{1-\sin^2(\theta)}\cos(\theta)\,d\theta \\ &= \int \sin^3(\theta)\cos^2(\theta)\,d\theta \\ &= \int \sin(\theta)\cos^2(\theta) - \sin(\theta)\cos^4(\theta)\,d\theta \\ &= -\frac{1}{3}\cos^3(\theta) + \frac{1}{5}\cos^5(\theta) + C.\end{aligned}$$

But

$$\cos(\theta) = \cos(\arcsin(x-1)) = \sqrt{1-(x-1)^2} = \sqrt{2x-x^2},$$

so

$$\int (x-1)^3\sqrt{2x-x^2}\,dx = \frac{1}{5}\sqrt{2x-x^2}^5 - \frac{1}{3}\sqrt{2x-x^2}^3 + C.$$

7. **Integration by Parts**

Compute:

(a) $\displaystyle\int x\arctan(x^2)\,dx =$

Solution:

$$\begin{aligned}\int x\arctan(x^2)\,dx &= \frac{x^2}{2}\arctan(x^2) - \int \frac{x^2}{2}\frac{2x}{1+x^4}\,dx \\ &= \frac{x^2}{2}\arctan(x^2) - \int \frac{x^3}{1+x^4}\,dx \\ &= \frac{x^2\arctan(x^2)}{2} - \frac{1}{4}\ln|1+x^4| + C.\end{aligned}$$

(b) $\int \sin(2x)\cos(x-1)\,dx =$

Solution:

$$\int \sin(2x)\cos(x-1)\,dx = \sin(2x)\sin(x-1) - \int 2\cos(2x)\sin(x-1)\,dx$$
$$\int \cos(2x)\sin(x-1)\,dx = -\cos(2x)\cos(x-1) - \int 2\sin(2x)\cos(x-1)\,dx$$
$$\int \sin(2x)\cos(x-1)\,dx = \sin(2x)\sin(x-1) + 2\cos(2x)\cos(x-1) + 4\int \sin(2x)\cos(x-1)\,dx$$
$$-3\int \sin(2x)\cos(x-1)\,dx = \sin(2x)\sin(x-1) + 2\cos(2x)\cos(x-1)$$
$$\int \sin(2x)\cos(x-1)\,dx = \frac{-1}{3}\sin(2x)\sin(x-1) + \frac{-2}{3}\cos(2x)\cos(x-1) + C.$$

4. **Integrals Involving Exponentials and Logarithms**

Compute the following integrals:

(a) $\int \frac{e^x}{1+e^x}\,dx =$

Solution: We take $u = 1+e^x$ so $du = e^x\,dx$. Then

$$\int \frac{e^x}{1+e^x}\,dx = \int \frac{1}{u}\,du$$
$$= \ln|u| + C = \ln|1+e^x| + C.$$

(b) $\int_0^2 x3^{x^2}\,dx =$

Solution:

Take $u = x^2$ so $du = 2x\,dx$ and $u(0)=0, u(2)=4$. Then

$$\int_0^2 x3^{x^2}\,dx = \int_0^4 \frac{1}{2}3^u\,du = \frac{3^u}{2\ln(3)}\Big|_0^4 = \frac{81-1}{2\ln(3)} = \frac{40}{\ln(3)}.$$

(c) $\int \frac{1}{2x + x\ln(x)}\,dx =$

Solution:

We can take $u = 2+\ln(x)$ so that $du = 1/x\,dx$. Then

$$\int \frac{1}{2x + x\ln(x)}\,dx = \int \frac{1}{u}\,du$$
$$= \ln|u| + C = \ln|2+\ln(x)| + C.$$