Math 1232 Sprin 2021
Single-Variable Calculus II Mastery Quiz 7
Due Friday, March 12

This week's mastery quiz has seven topics. You should do topics 12 and 11, and optionally one of the previous topics. Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 20-30 minutes on this quiz.

Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

14. Sequences
13. Differential Equations
12. Arc Length and Surface Area
11. Improper Integrals
10. Numeric Integration
9. Partial Fractions
7. Integration by parts

14. **Sequences**

(a) Consider the sequence $(a_n) = (6, 2, {}^{2}\!/_{3}, {}^{2}\!/_{9}, \dots)$. Find a formula for the nth term a_n. Compute $\lim_{n\to\infty} a_n$.

Solution: $a_n = \frac{18}{3^n}$ or $a_n = 2 \cdot 3^{2-n}$. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{18}{3^n} = 0.$$

We can justify this by saying the bottom goes to infinity.

(b) Let $b_n = \frac{3+5n^2}{1+n+n^2}$. Compute the first four terms of the sequence, and compute $\lim_{n\to\infty} b_n$, with justification.

Solution: $b_1 = \frac{8}{3}, b_2 = \frac{23}{7}, b_3 = \frac{48}{13}, b_4 = \frac{83}{21}$. If $f(x) = \frac{3+5x^2}{1+x+x^2}$ then the top and bottom both go to infinity, so by L'Hospital's Rule we have

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{3+5x^2}{1+x+x^2} = \lim_{x\to\infty} \frac{10x}{1+2x} = \lim_{x\to\infty} \frac{10}{2} = 5$$

and thus $\lim_{n\to\infty} b_n = 5$. Alternatively, we could see that

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{3+5n^2}{1+n+n^2} = \lim_{n\to\infty} \frac{3/n^2+5}{1/n^2+1/n+1} = \frac{0+5}{0+0+1} = 5.$$

(c) Let $c_n = \frac{(2n-1)!}{(2n+1)!}$. Compute the first three terms of this sequence, and compute $\lim_{n\to\infty} c_n$, with justification.

Solution:

$c_1 = \frac{1}{6}, c_2 = \frac{6}{120}, c_3 = \frac{120}{5040}$. We have

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{(2n-1)!}{(2n+1)!} = \lim_{n\to\infty} \frac{1}{(2n)(2n+1)} = 0,$$

since the bottom goes to infinity.

13. **Differential Equations**

(a) Find a general solution to the equation $xy' = y^2 + 1$.

Solution:

$$\begin{aligned} \frac{dy}{y^2+1} &= \frac{dx}{x} \\ \arctan(y) &= \ln(x) + C \\ y &= \tan(\ln(x) + C). \end{aligned}$$

(b) Find a (specific) solution to the initial value problem $y'/x - y = 1$ if $y(0) = 3$

Solution:

$$\begin{aligned}
y'/x &= 1+y \\
\frac{dy}{1+y} &= x\,dx \\
\ln|1+y| x^2/2 &+ C \\
1+y &= e^{x^2/2}e^C \\
y &= Ke^{x^2/2}-1 \\
3 &= K-1 \Rightarrow K=4 \\
y &= 4e^{x^2/2}-1.
\end{aligned}$$

12. **Arc Length and Surface Area**

   (a) Set up (but don't compute!) the arc length of the curve $\arctan(x) = y^3$ as x varies from 1 to 6.

   Solution: We have $y = \sqrt[3]{\arctan(x)}$ so $y' = \frac{1}{3}(\arctan(x))^{-2/3}\frac{1}{1+x^2}$, and thus

$$L = \int_1^6 \sqrt{1 + \frac{1}{9\arctan(x)^{4/3}(1+x^2)^2}}\,dx.$$

   (b) Compute the area of the surface obtained by taking the curve $y = \sqrt{15-x}$ as x goes from 3 to 5 and rotating it around the x-axis.

   Solution: We have $y' = \frac{-1}{2\sqrt{15-x}}$. So we get

$$\begin{aligned}
A &= \int_3^5 2\pi y\sqrt{1+y'^2}\,dx \\
&= \int_3^5 2\pi\sqrt{15-x}\sqrt{1+\frac{1}{4(15-x)}}\,dx \\
&= 2\pi\int_3^5 \sqrt{15-x+\frac{1}{4}}\,dx \\
&= \pi\int_3^5 \sqrt{61-4x}\,dx \\
&= \pi\frac{2}{3\cdot(-4)}(61-4x)^{3/2}\Big|_3^5 = \frac{-\pi}{6}\left(41^{3/2}-49^{3/2}\right) \\
&= \frac{\pi}{6}(343-41\sqrt{41}) =\approx 42.13.
\end{aligned}$$

11. **Improper Integrals**

   (a) Compute $\displaystyle\int_1^2 \frac{dx}{x\ln(x)} =$

   Solution:

$$
\begin{aligned}
\int_0^1 \frac{dx}{x\ln(x)} &= \lim_{t\to 1^+}\int_s^t \frac{dx}{x\ln(x)}\\
&= \lim_{t\to 1^+} \ln(|\ln(x)|)\Big|_t^2\\
&= \lim_{t\to 1^+} \ln(|\ln(2)|) - \ln|\ln(t)|
\end{aligned}
$$

But $\lim_{t\to 1^+}\ln(t) = 0$, so $\lim_{t\to 1^+}\ln|\ln(t)| = -\infty$. So this limit diverges.

(b) Compute $\displaystyle\int_1^\infty \frac{1}{x^4}\,dx =$

Solution:

$$
\begin{aligned}
\int_1^\infty \frac{dx}{x^4} &= \lim_{t\to\infty}\int_1^t \frac{dx}{x^4}\\
&= \lim_{t\to\infty} \frac{-1}{5x^5}\Big|_1^t\\
&= \lim_{t\to\infty} \frac{1}{5} - \frac{1}{5t^5} = \frac{1}{5}.
\end{aligned}
$$

10. **Numeric Integration**

(a) How many intervals do you need with the **midpoint** rule to approximate $\int_2^4 \frac{1}{x}\,dx$ to within 1/100? Compute that approximation. (Feel free to use a calculator to plug in numeric values, or to leave the answer in exact unsimplified terms, but show every step.)

Solution:

We have

$$
\begin{aligned}
f'(x) &- 1/x^2\\
f''(x) &= 2/x^3 f''(2) \qquad\qquad = 1/4\\
|E_M| &\le \frac{1/4\cdot 2^3}{24\cdot n^2} \le \frac{1}{100}\\
n^2 &\ge 100/12 = 25/3 \approx 8.33\\
n &\ge 4
\end{aligned}
$$

so we need to use at least four intervals. Then the midpoint approximation would be

$$
\begin{aligned}
\int_2^4 \frac{1}{x}\,dx &\approx \frac{1}{2}\left(\frac{1}{2.25}+\frac{1}{2.75}+\frac{1}{3.25}+\frac{1}{3.75}\right)\\
&= \frac{2}{9}+\frac{2}{11}+\frac{2}{13}+\frac{2}{15} = \frac{4448}{6435} \approx .6912
\end{aligned}
$$

The true answer is approximately .6931 so we're well within 1/100.

(b) Suppose we have

$$g(-3) = 2 \quad g(-2) = 4 \quad g(-1) = 3 \quad g(0) = 5 \quad g(1) = 8 \quad g(2) = 6 \quad g(3) = 1$$

Approximate $\int_{-3}^{3} g(x)\,dx$ using the trapezoid rule and using Simpson's rule.
Solution:
For the trapezoid rule, we have

$$\begin{aligned}
T_6 &= \frac{g(-3)+g(-2)}{2} + \frac{g(-2)+g(-1)}{2} + \frac{g(-1)+g(0)}{2} + \frac{g(0)+g(1)}{2} + \frac{g(1)+g(2)}{2} + \frac{g(2)+g(}{2} \\
&= \frac{6}{2} + \frac{7}{2} + \frac{8}{2} + \frac{13}{2} + \frac{14}{2} + \frac{7}{2} \\
&= \frac{55}{2} = 27.5.
\end{aligned}$$

For Simpson's rule, we have

$$\begin{aligned}
S_6 &= \frac{1}{3}\left(g(-3) + 4g(-2) + 2g(-1) + 4g(0) + 2g(1) + 4g(2) + g(3)\right) \\
&= \frac{1}{3}(2 + 16 + 6 + 20 + 16 + 24 + 1) \\
&= \frac{85}{3} \approx 28.33.
\end{aligned}$$

9. **Partial Fractions**

(a) Compute $\int \frac{x^3-x+2}{x+1}\,dx =$
Solution: Polynomial long division gives that

$$\begin{aligned}
x^3 - x + 2 &= x^2(x+1) - x(x+1) + 2 \\
\frac{x^3-x+2}{x+1} &= x^2 - x + \frac{2}{x+1} \\
\int \frac{x^3-x+2}{x+1}\,dx &= \int x^2 - x + \frac{2}{x+1}\,dx \\
&= \frac{x^3}{3} - \frac{x^2}{2} + 2\ln|x+1| + C.
\end{aligned}$$

(b) Compute $\int \frac{1}{x^2+5x+6}\,dx =$
Solution:

$$\begin{aligned}
\frac{1}{x^2+5x+6} &= \frac{1}{(x+2)(x+3)} \\
&= \frac{A}{x+2} + \frac{B}{x+3} \\
1 &= A(x+3) + B(x+2) \\
1 &= -B \Rightarrow B = -1 \\
1 &= A \Rightarrow A = 1 \\
\frac{1}{x^2+5x+6} &= \frac{1}{x+3} - \frac{1}{x+2} \\
\int \frac{1}{x^2+5x+6}\,dx &= \int \frac{1}{x+3} - \frac{1}{x+2}\,dx = \ln|x+3| - \ln|x+2| + C.
\end{aligned}$$

7. **Integration by Parts**

Use integration by parts to compute:

(a) $\displaystyle\int \sin(x)\ln(\cos(x))\,dx =$

Solution:

$$\begin{aligned}\int \sin(x)\ln(\cos(x))\,dx &= -\cos(x)\ln(\cos(x)) - \int (-\cos(x))(-\tan(x))\,dx\\ &= -\cos(x)\ln(\cos(x)) - \int \sin(x)\,dx\\ &= -\cos(x)\ln(\cos(x)) + \cos(x) + C.\end{aligned}$$

(b) $\displaystyle\int \frac{\sin(x)}{e^x}\,dx =$

Solution:

$$\begin{aligned}\int \frac{\sin(x)}{e^x}\,dx &= \int \sin(x)e^{-x}\,dx\\ &= -\sin(x)e^{-x} - \int -\cos(x)e^{-x}\,dx\\ \int \cos(x)e^{-x}\,dx &= -\cos(x)e^{-x} - \int (-\sin(x))(-e^{-x})\,dx\\ \int \frac{\sin(x)}{e^x}\,dx &= -\sin(x)e^{-x} - \cos(x)e^{-x} - \int \sin(x)e^{-x}\,dx\\ 2\int \frac{\sin(x)}{e^x}\,dx &= -\sin(x)e^{-x} - \cos(x)e^{-x} + C\\ \int \frac{\sin(x)}{e^x}\,dx &= -\frac{\sin(x)e^{-x} + \cos(x)e^{-x}}{2} + C.\end{aligned}$$