Math 1232 Sprin 2021
Single-Variable Calculus II Mastery Quiz 7
Due Friday, March 12

This week's mastery quiz has nine topics. You should do topics 16 and 15, and optionally *one* of the previous topics. Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 20-30 minutes on this quiz.

Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file*. You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

16. Divergence and Integral tests
15. Geometric and Telescoping Series
14. Sequences
13. Differential Equations
12. Arc Length and Surface Area
11. Improper Integrals
10. Numeric Integration
8. Trigonometric Integrals
6. L'Hospital's Rule

16. **Divergence and Integral Tests**

Determine whether each of the following series converges or diverges. Justify your answers using only the divergence and integral tests (and *not* the comparison tests).

(a) $\displaystyle\sum_{i=1}^{\infty} \frac{n}{n^4+1}$

Solution: We compute

$$\begin{aligned}\int_1^\infty \frac{x}{x^4+1}\,dx &= \lim_{t\to\infty}\int_1^t \frac{x}{x^4+1}\,dx \\ &= \lim_{t\to\infty}\frac{1}{2}\arctan(x^2)\Big|_1^t = \lim_{t\to\infty}\frac{1}{2}\left(\arctan(t^2)-\arctan(1)\right) \\ &= \frac{1}{2}\left(\pi/2-\pi/4\right) = \frac{\pi}{8}.\end{aligned}$$

This is finite and convergent, so by the integral test, the series $\sum_{i=1}^{\infty}\frac{n}{n^4+1}$ converges.

(b) $\displaystyle\sum_{i=1}^{\infty} \frac{n}{\ln(n)}$

Solution: By L'Hospital's rule, we compute that

$$\lim_{n\to\infty}\frac{n}{\ln(n)} = \lim_{x\to\infty}\frac{1}{1/x} = \lim_{x\to\infty} x = \infty.$$

Since this isn't zero, the series diverges by the divergence test.

(c) $\displaystyle\sum_{i=1}^{\infty} \frac{4n^3+1}{n^4+n+3}$

Solution: We have

$$\begin{aligned}\int_1^\infty \frac{4x^3+1}{x^4+x+3}\,dx = \lim_{t\to\infty}\int_1^t \frac{4x^3+1}{x^4+x+3}\,dx &= \lim_{t\to\infty}\ln(x^4+x+3)\Big|_1^t \\ &= \lim_{t\to\infty}\ln(t^4+t+3)-\ln(5) = \infty.\end{aligned}$$

This diverges, so by the integral test the series $\sum_{i=1}^{\infty}\frac{4n^3+1}{n^4+n+3}$ diverges.

15. **Geometric and Telescoping Series**

Compute the following infinite sums, with justification:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+5n+6} =$

Solution:

A partial fractions decomposition tells us that $\frac{2}{n^2+5n+6} = \frac{2}{n+2} - \frac{2}{n+3}$. Then our partial sums are

$$\sum_{n=1}^{k}\frac{2}{n^2+5n+6} = \left(\frac{2}{3}-\frac{2}{4}\right)+\left(\frac{2}{4}-\frac{2}{5}\right)+\cdots+\left(\frac{2}{k+2}-\frac{2}{k+3}\right) = \frac{2}{3}-\frac{2}{k+3}$$

$$\sum_{n=1}^{\infty}\frac{2}{n^2+5n+6} = \lim_{k\to\infty}\frac{2}{n^2+5n+6} = \lim_{k\to\infty}\frac{2}{3}-\frac{2}{k+3} = \frac{2}{3}.$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{4}{3^{2n}} =$

Solution: This is a geometric series with $a = 4/9$ and $r = 1/9$, so we have

$$\sum_{n=1}^{\infty} \frac{4}{3^{2n}} == \frac{4/9}{1-1/9} = 1/2.$$

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{3^{2n}}{5^n} =$

Solution:

This is a geometric series with $a = 9/5$ and $r = 9/5$. We have $r > 1$, so we know that it diverges.

14. **Sequences**

(a) Consider the sequence $(a_n) = ({}^1\!/\!{}_4, {}^2\!/\!{}_5, {}^3\!/\!{}_6, \ldots)$. Find a formula for the nth term a_n. Compute $\lim_{n\to\infty} a_n$.

Solution:

$a_n = \frac{n}{n+3}$. We have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{1+3/n} = 1.$$

(b) Let $b_n = \frac{(n)!}{(n+2)!}$. Compute the first four terms of the sequence, and compute $\lim_{n\to\infty} b_n$, with justification.

Solution:

$b_1 = 1/6, b_2 = 2/24 = 1/12, b_3 = 6/120 = 1/20$, and $b_4 = 24/720 = 1/30$.

We compute

$$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{(n+1)(n+2)} = \lim_{n\to\infty} \frac{1}{n^2+3n+2} = 0.$$

(c) Let $c_n = \frac{3n+2}{4n-1}$. Compute the first four terms of this sequence, and compute $\lim_{n\to\infty} c_n$, with justification.

Solution:

$c_1 = 5/3, c_2 = 8/7, c_3 = 11/11 = 1$, and $c_4 = 14/15$. We have

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} \frac{3+2/n}{4-1/n} = \frac{3}{4}.$$

13. **Differential Equations**

(a) Find a general solution to the equation $y' = x^2 + 1 + x^2y + y$.

Solution:

$$\frac{dy}{dx} = (x^2+1)(y)$$
$$\frac{dy}{y} = x^2 + 1\,dx \ln|y| \qquad = x^3/3 + x + C$$
$$y = e^{x^3/3+x+C}.$$

(b) Find a (specific) solution to the initial value problem $-2x + 4y^3\sqrt{x^2+4}y' = 0$ if $y(0) = 2$

Solution:

$$\begin{aligned}
4y^3y'\sqrt{x^2+4} &= 2x \\
4y^3\,dy &= \frac{2x}{\sqrt{x^2+4}}\,dx \\
y^4 &= 2\sqrt{x^2+4} + C \\
y &= \sqrt[4]{2\sqrt{x^2+4}+C}.
\end{aligned}$$

Then we have

$$\begin{aligned}
2 &= \sqrt[4]{2\sqrt{4}+C} = \sqrt[4]{4+C} \\
16 &= 4 + C \\
C &= 12 \\
y &= \sqrt[4]{2\sqrt{x^2+4}+12}.
\end{aligned}$$

12. **Arc Length and Surface Area**

(a) Compute the arc length of the curve $y = \frac{x^4}{8} + \frac{1}{4x^2}$ as x varies from 2 to 4.

Solution: We have $y' = \frac{x^3}{2} + \frac{-1}{2x^3}$, and thus

$$\begin{aligned}
L &= \int_1^2 \sqrt{1 + (x^3/2 - x^{-3}/2)}\,dx = \int_1^2 \sqrt{1 + x^6/4 - 1/2 + x^{-6}/4}\,dx \\
&= \int_1^2 \sqrt{x^6/4 + 1/2 + x^{-6}/4}\,dx = \int_1^2 \sqrt{(x^3/2 + x^{-3}/2)^2}\,dx \\
&= \int_1^2 x^3/2 + x^{-3}/2\,dx = \frac{x^4}{8} - \frac{1}{4x^2}\Big|_1^2 \\
&= \frac{16}{8} - \frac{1}{16} - \left(\frac{1}{8} - \frac{1}{4}\right) = \frac{33}{16}.
\end{aligned}$$

(b) Set up (but don't compute!) the area of the surface obtained by taking the curve $x = \ln y^2 + 5y$ as y goes from 2 to 7 and rotating it around the y-axis.

Solution: We have $x' = \frac{2y+5}{y^2+5y}$ so

$$L = \int_2^7 2\pi \ln(y^2+5y)\sqrt{1 + \left(\frac{2y+5}{y^2+5y}\right)^2}\,dy.$$

11. **Improper Integrals**

(a) Compute $\displaystyle\int_0^{\pi/2} \frac{\sin(t)}{\sqrt{\cos(t)}}\,dt.$

Solution: We know that $\frac{\sin(t)}{\sqrt{\cos(t)}}$ is undefined at $\pi/2$. So we have

$$\begin{aligned}\int_0^{\pi/2} \frac{\sin(t)}{\sqrt{\cos(t)}}\,dt &= \lim_{t\to\pi/2}\int_0^t \frac{\sin(t)}{\sqrt{\cos(t)}}\,dt \\ &= \lim_{t\to\pi/2} -2\sqrt{\cos(t)}\Big|_0^t \\ &= \lim_{t\to\pi/2} -2\sqrt{\cos(t)}+2 = 2.\end{aligned}$$

(b) Compute $\displaystyle\int_1^\infty \frac{dx}{\sqrt{x}+x\sqrt{x}}$.

Solution: We'll take $u=\sqrt{x}$ so $du = \frac{dx}{2\sqrt{x}}$ and $\frac{dx}{\sqrt{x}+x\sqrt{x}} = \frac{2du}{1+u^2}$. Then

$$\begin{aligned}\int_1^\infty \frac{dx}{\sqrt{x}+x\sqrt{x}} &= \lim_{t\to\infty}\int_1^t \frac{dx}{\sqrt{x}+x\sqrt{x}} \\ &= \lim_{t\to\infty}\int_1^{\sqrt{t}} \frac{2du}{1+u^2} \\ &= \lim_{t\to\infty} 2\arctan(u)\Big|_1^{\sqrt{t}} = \lim_{t\to\infty} 2\arctan(\sqrt{t}) - 2\arctan(1) \\ &= 2(\pi/2)-2(\pi/4) = \pi/2.\end{aligned}$$

10. **Numeric Integration**

(a) How many intervals do you need with **Simpson's** rule to approximate $\int_2^4 \frac{3}{x}$ to within 1/500? Compute that approximation. (Feel free to use a calculator to plug in numeric values, or to leave the answer in exact unsimplified terms, but show every step.)

Solution:

We have $f'(x) = -3/x^2, f''(x) = 6/x^3, f'''(x) = -18/x^4, f^{(4)}(x) = 72/x^5$, and thus the maximum possible value of $f^{(4)}(x)$ is $f^{(4)}(2) = \frac{72}{32} = \frac{9}{4}$. Then we have

$$|E_S| \le \frac{{}^9\!/\!_4 \cdot 2^5}{180n^4} = \frac{72}{180n^4} = \frac{2}{5n^4}.$$

We want $1/500 \ge \frac{2}{5n^4}$ which implies

$$\begin{aligned} n^4 &\ge \frac{1000}{5} = 200 \\ n &> 3 \end{aligned}$$

So we need n to be at least 4.

Then we have

$$\begin{aligned} S_4 &= \frac{1}{6}\left(f(2)+4f(5/2)+2f(3)+4f(7/2)+f(4)\right) \\ &= \frac{1}{6}\left(\frac{3}{2}+\frac{24}{5}+\frac{6}{3}+\frac{24}{7}+\frac{3}{4}\right) = \frac{1747}{840} \approx 2.07976. \end{aligned}$$

Since the true answer is 2.07944, this is pretty good.

(b) Suppose we have

$$g(4) = 2 \quad g(5) = 4 \quad g(6) = 6 \quad g(7) = 4 \quad g(8) = 6 \quad g(9) = 7 \quad g(10) = 2$$

Approximate $\int_4^{10} g(x)\,dx$ using the trapezoid rule and the midpoint rule.

Solution:

For the trapezoid rule, we have

$$\begin{aligned} T_6 &= \frac{g(4)+g(5)}{2} + \frac{g(5)+g(6)}{2} + \frac{g(6)+g(7)}{2} + \frac{g(7)+g(8)}{2} + \frac{g(8)+g(9)}{2} + \frac{g(9)+g(10)}{2} \\ &= \frac{6}{2} + \frac{10}{2} + \frac{10}{2} + \frac{10}{2} + \frac{13}{2} + \frac{9}{2} \\ &= \frac{58}{2} = 29. \end{aligned}$$

For the midpoint rule, we can only find three midpoints. So we have

$$M_3 = 2g(5) + 2g(7) + 2g(9) = 8 + 8 + 14 = 30.$$

8. **Trigonometric Integrals**

Compute

(a) $\displaystyle\int \sec^2(t)\tan^3(t)\,dt =$

Solution:

We're going to take $u = \tan(t)$ so that $du = \sec^2(t)\,dt$. Then

$$\int \sec^2(t)\tan^3(t)\,dt = \int u^3\,du = \frac{u^4}{4} + C = \frac{1}{4}\tan^4(t) + C.$$

(b) $\displaystyle\int_0^{1/\sqrt{3}} \frac{x^3}{\sqrt{1+x^2}}\,dx =$

Solution:

We'll set $x^2 = \tan^2(\theta)$ so $x = \tan(\theta)$, and then $dx = \sec^2(\theta)\,d\theta$. As x goes from 0 to $1/\sqrt{3}$ we have $\tan\theta$ goign from 0 to $1/\sqrt{3}$, and thus θ going from 0 to $\pi/6$. Then we have

$$\begin{aligned} \int_0^{1/\sqrt{3}} \frac{x^3}{\sqrt{1+x^2}}\,dx &= \int_0^{\pi/6} \frac{\tan^3(\theta)}{\sqrt{1+\tan^2(\theta)}}\sec^2(\theta)\,d\theta \\ &= \int_0^{\pi/6} \tan^3(\theta)\sec(\theta)\,d\theta \\ &= \int_0^{\pi/6} \tan(\theta)\sec(\theta)(\sec^2(\theta)-1)\,d\theta \\ &= \frac{1}{3}\sec^3(\theta) - \sec(\theta)\Big|_0^{\pi/6} \\ &= \frac{8}{9\sqrt{3}} - \frac{2}{\sqrt{3}} - \left(\frac{1}{3} - 1\right) = \frac{2}{3} - \frac{10}{9\sqrt{3}}. \end{aligned}$$

6. **L'Hospital's Rule**

Compute the following limits:

(a) $\lim_{x\to\infty} \frac{x\ln(x)}{x^2}$
Solution:

$$\begin{aligned}\lim_{x\to\infty} \frac{x\ln(x)}{x^2} &= \lim_{x\to\infty} \frac{\ln(x)+1}{2x}\\ &= \lim_{x\to\infty} \frac{1/x}{2} = 0.\end{aligned}$$

(b) $\lim_{x\to 0} \frac{\arctan(x)}{\ln(x+1)} =$
Solution:

$$\lim_{x\to 0} \frac{\arctan(x)}{\ln(x+1)} = \lim_{x\to 0} \frac{1/(x^2+1)}{1/(x+1)} = \lim_{x\to 0} \frac{x+1}{x^2+1} = 1.$$

(c) $\lim_{x\to\infty} x^{\ln(3)/(2+\ln(x))} =$
Solution:

$$\begin{aligned}\ln(y) &= \frac{\ln(3)\ln(x)}{2+\ln(x)}\\ \lim_{x\to\infty} \frac{\ln(3)\ln(x)}{2+\ln(x)} &= \lim_{x\to\infty} \frac{\ln(3)/x}{1/x} = \ln(3)\\ \lim_{x\to\infty} y &= e^{\ln(3)} = 3.\end{aligned}$$