# Math 1232 Sprin 2021
# Single-Variable Calculus II Mastery Quiz 7
# Due Friday, March 12

This week's mastery quiz has eleven topics. You should do topics 16 and 15, and optionally *one* of the previous topics. Don't worry if you make a minor error, but try to demonstrate your mastery of the underlying material. You shouldn't spend more than 20-30 minutes on this quiz.

Feel free to consult your notes, but please don't talk about the actual quiz questions with other students in the course.

Remember that you are trying to demonstrate that you understand the concepts involved. For all these problems, justify your answers and explain how you reached them. Do not just write "yes" or "no" or give a single number.

Please upload your work as *one PDF file.* You can produce the file on your computer/tablet/whatever, or you can handwrite it and then scan it. If you have a smartphone, there are many apps that can help you produce a clean single pdf; I personally have used GeniusScan but there are many options.

18. Absolute and Conditional Convergence
17. Comparison Test and Limit Comparison Test
16. Divergence and Integral tests
15. Geometric and Telescoping Series
14. Sequences
13. Differential Equations
12. Arc Length and Surface Area
11. Improper Integrals
10. Numeric Integration
6. L'Hospital's Rule
5. Inverse Trigonometric Functions

18. **Absolute and Conditional Convergence**

For each series, tell whether it absolutely converges, conditionally converges, or diverges. Justify your answer (and in particular, if it conditionally converges, explain why it doesn't absolutely converge).

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n n}{n^2+1}$

**Solution:**

This is an alternating series. Since the terms $\frac{n}{n^2+1}$ tend to zero as $n$ goes to infinity, this converges by the alternating series test.

However, it doesn't absolutely converge. If we look at the absolute value series, we have $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$. You can see this doesn't converge in a couple ways. The integral test would work. The regular comparison test will *not* work unless you're really careful, but $\frac{n}{n^2+1} < \frac{1}{n}$ so we'd need to do some chicanery.

So it seems like this calls for the limit comparison test. We have

$$\lim_{n\to\infty} \frac{n/n^2+1}{1/n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1.$$

Since the harmonic series $\sum \frac{1}{n}$ diverges, by the limit comparison test, $\sum \frac{n}{n^2+1}$ diverges, and thus our series does not converge absolutely.

(b) $\sum_{n=1}^{\infty} \frac{(-1)^n 3^n}{5^n+1}$

**Solution:**

We use the Ratio test. We have

$$\begin{aligned}\lim_{n\to\infty} \left| \frac{(-1)^{n+1}3^{n+1}/5^{n+1}+1}{(-1)^n 3^n/5^n+1} \right| &= \lim_{n\to\infty} \frac{3^{n+1}(5^n+1)}{3^n(5^{n+1}+1)} \\ &= \lim_{n\to\infty} 3\frac{5^n+1}{5^{n+1}+1} \\ &= \lim_{n\to\infty} 3\frac{1+1/5^n}{5+1/5^n} = \frac{3}{5}.\end{aligned}$$

This limit is less than 1, so by the ratio test this converges absolutely.

(c) $\sum_{n=1}^{\infty} (-1)^n \left(\frac{2n^2+n+1}{n^2-3n+2}\right)^n$

**Solution:** We use the root test. We have

$$\lim_{n\to\infty} \sqrt[n]{\left|(-1)^n \left(\frac{2n^2+n+1}{n^2-3n+2}\right)^n\right|} = \lim_{n\to\infty} \frac{2n^2+n+1}{n^2-3n+2} = 2 > 1.$$

So by the root test, this series diverges.

17. **Comparison Tests**

Determine whether each of the following series converges by using an appropriate comparison test.

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{n\sin^2(n)}{n^3+2}$

**Solution:** You can't really use the limit comparison test here, because the $\sin(n)$ will screw it up. But you can use the usual comparison test. We know that $0 \leq \sin^2(n) \leq 1$, so

$$\frac{n\sin^2(n)}{n^3+2} \leq \frac{n}{n^3} = \frac{1}{n^2}.$$

We know that $\sum \frac{1}{n^2}$ converges by the $p$-series test, so our series converges by the comparison test.

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2-3}{n^3+2}$

**Solution:**

Here it would be hard to use the regular comparison test. It's true that $\frac{n^2-3}{3n^3+2} \leq \frac{1}{n}$, but since this says it's less than a divergent series, it doesn't really help.

Instead, we limit compare to $\frac{1}{n}$. We have

$$\lim_{n\to\infty} \frac{n^2-3/3n^3+1}{1/n} = \lim_{n\to\infty} \frac{n^3-3n}{3n^3+1} = \frac{1}{3}.$$

Since this is a finite non-zero number, the two series have the same convergence behavior. (It doesn't matter that $1/3 < 1$; all that matters is it's finite and non-zero.) Thus, since $\frac{1}{n}$ diverges, we know that our series also diverges.

16. **Divergence and Integral Tests**

Determine whether each of the following series converges or diverges. Justify your answers using only the divergence and integral tests (and *not* the comparison tests).

(a) $\displaystyle\sum_{n=1}^{\infty} \sqrt{n^3+1}$

**Solution:** There's no way we're going to successfully compute an integral here. But $\lim_{n\to\infty} \sqrt{n^3+1} = \infty$, so by the divergence test this series diverges.

(b) $\displaystyle\sum_{n=1}^{\infty} ne^{-n^2}$

**Solution:** We can work this out with the integral test. We have

$$\begin{aligned}\int_1^{\infty} xe^{-x^2}\,dx &= \lim_{t\to\infty} \int_1^t xe^{-x^2}\,dx \\ &= \lim_{t\to\infty} \frac{-1}{2} e^{-x^2}\Big|_1^t \\ &= \lim_{t\to\infty} \frac{1}{2e} - \frac{1}{2e^{t^2}} = \frac{1}{2e} < \infty.\end{aligned}$$

Since this integral converges, the series must also converge by the integral test.

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n}}{\sqrt{n^3}+1}$

**Solution:** We take $u = \sqrt{x^3}+1$ so that $du = \frac{3}{2}\sqrt{x}\,dx$, and then

$$\begin{aligned}\int_1^{\infty} \frac{\sqrt{x}}{\sqrt{x^3}+1}\,dx &= \lim_{t\to\infty}\int_1^t \frac{\sqrt{x}}{\sqrt{x^3}+1}\,dx \\ &= \lim_{t\to\infty}\frac{2}{3}\ln(\sqrt{x^3}+1)\Big|_1^t \\ &= \lim_{t\to\infty}\frac{2}{3}\Big(\ln(\sqrt{t^3}+1)-\ln(2)\Big) = \infty.\end{aligned}$$

Since this improper integral is infinite, by the integral test the series does not converge.

15. **Geometric and Telescoping Series**

Compute the following infinite sums, with justification:

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{3\cdot 5^{n-1}} =$

**Solution:**

This is a geometric series with $a = \frac{2}{3}$ and $r = \frac{2}{5}$. So the sum is

$$\frac{2/3}{1-2/5} = \frac{2/3}{3/5} = \frac{10}{9}.$$

(b) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} =$

**Solution:**

We can do a partial fractions decomposition: we have

$$\begin{aligned}2 &= A(n+1)+B(n+3) \\ 2 &= 2B && \Rightarrow B = 1 \\ 2 &= -2A && \Rightarrow A = -1\end{aligned}$$

so our sum is

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{1}{n+1} - \frac{1}{n+3} &= \left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right)+\dots \\ &= \frac{1}{2}+\frac{1}{3} = \frac{5}{6}.\end{aligned}$$

More rigorously, we have

$$\begin{aligned}
\sum_{n=1}^{k} \frac{2}{n^2+4n+3} &= \left(\frac{1}{2}-\frac{1}{4}\right)+\left(\frac{1}{3}-\frac{1}{5}\right)+\left(\frac{1}{4}-\frac{1}{6}\right) \\
&\quad + \cdots + \left(\frac{1}{k+1}-\frac{1}{k+3}\right) \\
&= \frac{1}{2}+\frac{1}{3}-\frac{1}{k+2}-\frac{1}{k+3} \\
\sum_{n=1}^{\infty} \frac{2}{n^2+4n+3} &= \lim_{k\to\infty} \frac{1}{2}+\frac{1}{3}-\frac{1}{k+2}-\frac{1}{k+3} \\
&= \frac{1}{2}+\frac{1}{3} = \frac{5}{6}.
\end{aligned}$$

(c) $\sum_{n=1}^{\infty} \frac{2^{2n+1}}{3^n} =$

**Solution:**

This is a geometric series with $a = \frac{8}{3}$ and $r = \frac{4}{3}$. Since $r > 1$ this series does not converge.

14. **Sequences**

(a) Consider the sequence $(a_n) = \left(\frac{1}{2}, {}^{2}\!/\!{}_{5}, {}^{3}\!/\!{}_{10}, {}^{4}\!/\!{}_{17}, \dots\right)$. Find a formula for the $n$th term $a_n$. Compute $\lim_{n\to\infty} a_n$.

**Solution:**

We have $a_n = \frac{n}{n^2+1}$, and thus

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n}{n^2+1} = \lim_{n\to\infty} \frac{1/n}{1+1/n^2} = 0.$$

(b) Let $b_n = \frac{n!}{2^n}$. Compute the first four terms of the sequence, and compute $\lim_{n\to\infty} b_n$, with justification.

**Solution:**

$$b_1 = \frac{1}{2} \qquad b_2 = \frac{2}{4}$$
$$b_3 = \frac{6}{8} \qquad b_4 = \frac{24}{16}.$$

We see that

$$\frac{n!}{2^n} = \frac{n(n-1)(n-2)\dots(2)(1)}{2(2)(2)\dots(2)(2)} \geq \frac{n}{2}.$$

Since $\lim_{n\to\infty} \frac{n}{2} = \infty$, we know that $\lim_{n\to\infty} \frac{n!}{2^n} = \infty$.

(c) Let $c_n = \frac{\log_2(n)}{n}$. Compute the first four terms of this sequence, and compute $\lim_{n\to\infty} c_n$, with justification.

**Solution:**

$$c_1 = 0 \qquad c_2 = \frac{1}{2}$$
$$c_3 = \frac{\log_2 3}{3} \qquad c_4 = \frac{2}{4}$$

We can look at this as a function and use L'Hospital's rule:

$$\lim_{x\to\infty} \frac{\log_2(x)}{x} = \lim_{x\to\infty} \frac{1/x\ln(2)}{1} = \lim_{x\to\infty} \frac{1}{x\ln(2)} = 0.$$

Thus the limit of our sequence is zero.

13. **Differential Equations**

(a) Find a general solution to the equation $y' = xe^x y$.
**Solution:**

$$\frac{dy}{dx} = xye^y$$
$$\frac{dy}{y} = xe^x\,dx \ln|y| \qquad = xe^x - e^x + C$$
$$y = e^{xe^x - e^x}e^C.$$

(b) Find a (specific) solution to the initial value problem $y'/x = \cos^2(y)$ if $y(0) = \pi/3$
**Solution:**

$$\frac{dy}{dx} = x\cos^2(y)$$
$$\sec^2(y)\,dy = x\,dx$$
$$\tan(y) = x^2/2 + Cy \qquad = \arctan(x^2/2 + C).$$

Then we have

$$\pi/3 = \arctan(C)\tan(\pi/3) \qquad = CC = \sqrt{3}$$
$$y = \arctan(x^2/2 + \sqrt{3}).$$

12. **Arc Length and Surface Area**

(a) Set up (but don't compute!) an integral for the arc length of the curve $y = 1 + 3x + e^x$ as $y$ goes from 2 to $4 + e$.
**Solution:** We have $y' = 3 + e^x$. as $y$ goes from 2 to $4 + e$ we have $x$ going from 0 to 1, so we get

$$L = \int_0^1 \sqrt{1 + (3 + e^x)^2}\,dx.$$

We could try to solve this the other way and integrate it with respect to $y$, with integral bounds going from 2 to $4 + e$. But then we'd need to invert the function $f(x) = 1 + 3x + e^x$, and while this function is definitely invertible I can't really write down a formula for it.

(b) Compute the area of the surface obtained by taking the curve $x^{2/3} + y^{2/3} = 1$ as $x$ goes from 0 to 1 and rotating it around the $y$-axis.

**Solution:** We have $y = (1 - x^{2/3})^{3/2}$, so

$$\begin{aligned} y' &= \frac{3}{2}(1 - x^{2/3})^{1/2} \cdot \frac{-2}{3} x^{-1/3} \\ &= \sqrt{1 - x^{2/3}} x^{-1/3} \\ (y')^2 &= (1 - x^{2/3}) x^{-2/3} = x^{-2/3} - 1 \\ \sqrt{1 + (y')^2} &= \sqrt{x^{-2/3}} = x^{-1/3}. \end{aligned}$$

Then we compute

$$\begin{aligned} L &= \int_0^1 2\pi x \sqrt{1 + (y')^2}\, dx \\ &= 2\pi \int_0^1 x \cdot x^{-1/3}\, dx \\ &= 2\pi \int_0^1 x^{2/3}\, dx = 2\pi \frac{3}{5} x^{5/3} \Big|_0^1 \\ &= \frac{6\pi}{5} \left(1^{5/3} - 0^{5/3}\right) = \frac{6\pi}{5}. \end{aligned}$$

(If you graph the shape, you might notice that there's a symmetry, where we actually have $y = \pm(1 - x^{2/3})^{3/2}$. If you use both halves, you'll get twice this answer.)

11. **Improper Integrals**

(a) Compute $\displaystyle\int_{-1}^{1} \frac{1}{\sqrt[3]{x^2}}\, dx$.

**Solution:** We know that $\frac{1}{\sqrt[3]{x^2}}$ is undefined at zero. So we need to split this in half:

$$\begin{aligned} \int_{-1}^{1} \frac{1}{\sqrt[3]{x^2}}\, dx &= \int_{-1}^{0} \frac{1}{\sqrt[3]{x^2}}\, dx + \int_{0}^{1} \frac{1}{\sqrt[3]{x^2}}\, dx \\ &= \lim_{t \to 0^-} \int_{-1}^{t} x^{-2/3}\, dx + \lim_{s \to 0^+} \int_{s}^{1} x^{-2/3}\, dx \\ &= \lim_{t \to 0^-} 3x^{1/3} \Big|_{-1}^{t} + \lim_{x \to 0^+} 3x^{1/3} \Big|_{s}^{1} \\ &= 3 \lim_{t \to 0^-} \left( (\sqrt[3]{t} - \sqrt[3]{-1}) + (\sqrt[3]{1} - \sqrt[3]{s}) \right) \\ &= 3(0 + 1 + 1 - 0) = 6. \end{aligned}$$

(b) Compute $\displaystyle\int_{1}^{\infty} x e^{-x^2}\, dx$.

**Solution:** We'll take $u = -x^2$ so $du = -2x\,dx$. Then

$$\begin{aligned}\int_1^\infty xe^{-x^2}\,dx &= \lim_{t\to\infty}\int_1^t xe^{-x^2}\,dx \\ &= \lim_{t\to\infty}\int_{-1}^{-t^2}\frac{-1}{2}e^u\,du \\ &= \lim_{t\to\infty}\frac{-1}{2}e^u\Big|_{-1}^{-t^2} \\ &= \frac{-1}{2}\lim_{t\to\infty} e^{-t^2} - e^{-1} \\ &= \frac{-1}{2}(0-e^{-1}) = \frac{1}{2e}.\end{aligned}$$

10. **Numeric Integration**

(a) How many intervals do you need with the trapezoid rule to approximate $\int_0^\pi \sin(x)\,dx$ to within 1/4? Compute that approximation.

**Solution:**

We have $f'(x) = \cos(x)$ and $f''(x) = -\sin(x)$, so $|f''(x)| \leq 1$. Then

$$|E_T| \leq \frac{1\cdot\pi^3}{12n^2}$$

and since we want $|E_T| \leq \frac{1}{4}$ we have

$$\begin{aligned}\frac{1\cdot\pi^3}{12n^2} &\leq 4 \\ 4\pi^3 &\geq 12n^2 \\ n^2 &\geq \frac{\pi^3}{3} \approx 10\end{aligned}$$

so we need $n \geq 4$. Then the Trapezoid rule gives

$$\begin{aligned}&\int_0^\pi \sin(x)\,dx \\ &\approx \frac{\pi}{4}\frac{\sin(0)+\sin(\pi/4)}{2} + \frac{\pi}{4}\frac{\sin(\pi/4)+\sin(\pi/2)}{2} + \frac{\pi}{4}\frac{\sin(\pi/2)+\sin(3\pi/4)}{2} + \frac{\pi}{4}\frac{\sin(3\pi/4)+\sin(\pi)}{2} \\ &= \frac{\pi}{8}\left((0+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}+1+1+\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}+0\right) = \frac{\pi}{4}(1+\sqrt{2}) \approx 1.896.\end{aligned}$$

This is indeed within a quarter of the true answer of 2.

(b) Suppose we have

$$g(3) = 2 \quad g(5) = 5 \quad g(7) = 3 \quad g(9) = 7 \quad g(11) = 8 \quad g(13) = 9 \quad g(15) = 1$$

Approximate $\int_3^9 g(x)\,dx$ using the midpoint rule and Simpson's rule.

**Solution:**

For the midpoint rule, we have

$$M_3 = 4g(4) + 4g(6) + 4g(8) = 20 + 28 + 36 = 84.$$

For Simpson's rule, weh ave

$$\begin{aligned} S_6 &= \frac{2}{3}\left(2 + 4\cdot 5 + 2\cdot 3 + 4\cdot 7 + 2\cdot 8 + 4\cdot 9 + 1\right) \\ &= \frac{2}{3}(2 + 20 + 6 + 28 + 16 + 36 + 1) = \frac{2}{3}\cdot 109 = \frac{218}{3} \approx 72.67. \end{aligned}$$

6. **L'Hospital's Rule**

Compute the following limits:

(a) $\lim_{x\to\infty} \frac{1 + \ln(x) + (\ln(x))^2}{\sqrt{x}} =$

**Solution:** The top and bottom both approach infinity, so we can use L'Hospital's Rule.

$$\begin{aligned} \lim_{x\to\infty} \frac{1 + \ln(x) + (\ln(x))^2}{\sqrt{x}} &= \lim_{x\to\infty} \frac{1/x + 2\ln(x)/x}{1/2\sqrt{x}} = \lim_{x\to\infty} \frac{2\sqrt{x} + 4\sqrt{x}\ln(x)}{x} \\ &= \lim_{x\to\infty} \frac{2}{\sqrt{x}} + \frac{4\ln(x)}{\sqrt{x}} \\ &= 0 + \lim_{x\to\infty} \frac{4/x}{1/2\sqrt{x}} = \lim_{x\to\infty} \frac{8}{\sqrt{x}} = 0. \end{aligned}$$

(b) $\lim_{s\to 0} \frac{7^s - 3^{2s}}{s} =$

**Solution:**

The top and bottom both approach 0, so we can use L'Hospital's rule.

$$\lim_{s\to 0} \frac{7^s - 3^{2s}}{s} = \lim_{s\to 0} \frac{7^s \ln(7) - 3^{2s}\ln(3)\cdot 2}{1} = \ln(7) - 2\ln(3).$$

(c) $\lim_{x\to 0}(2x+1)^{\cot(x)} =$

**Solution:** Taking logs of both sides gives

$$\begin{aligned} y &= (2x+1)^{\cot(x)} \\ \ln|y| &= \cot(x)\ln|2x+1| = \frac{\cos(x)\ln|2x+1|}{\sin(x)}. \end{aligned}$$

The top and bottom both approach 0, so we can use L'Hospital's Rule:

$$\begin{aligned} \lim_{x\to 0}\ln|y| &= \lim_{x\to 0}\cos(x)\frac{\ln|2x+1|}{\sin(x)} = 1\cdot \lim_{x\to 0}\frac{\ln|2x+1|}{\sin(x)} \\ &= \lim_{x\to 0}\frac{2/(2x+1)}{\cos(x)} = 2 \\ \lim_{x\to 0} y &= e^2. \end{aligned}$$

5. **Inverse Trigonometric Functions**

(a) Compute $\arcsin\left(-\sqrt{3}/2\right) =$

**Solution:** $-\pi/3$ (and, importantly, not $5\pi/3$).

(b) Compute $\sin(\arccos(5/13))$.

**Solution:**

If we draw a triangle, we have adjacent side with length 5 and hypotenuse with length 13. By the Pythagorean theorem, the opposite side will have length $\sqrt{169-25} = 12$, and thus $\sin(\theta) = \frac{12}{13}$.

(c) $\dfrac{d}{dx}\arctan(x^3 + 1/x) =$

**Solution:**

$$\frac{d}{dx}\arctan(x^3 + 1/x) = \frac{1}{1 + (x^3 + 1/x)^2} \cdot (3x^2 - 1/x^2).$$

(d) $\displaystyle\int \frac{1}{\sqrt{9-4x^2}}\,dx.$

**Solution:**

We want $4x^2 = 9u^2$ so we have $u = 2x/3$ and $du = 2/3\,dx$. Then

$$\begin{aligned}\int \frac{1}{\sqrt{9-4x^2}}\,dx &= \int \frac{1}{\sqrt{9-9u^2}}\frac{3}{2}\,du \\ &= \int \frac{1}{2}\frac{1}{\sqrt{1-u^2}}\,du \\ &= \frac{1}{2}\arcsin(u) + C = \frac{1}{2}\arcsin(2x/3) + C.\end{aligned}$$