$|0\rangle, |1\rangle$

$|011\rangle = |0\rangle \otimes |1\rangle \otimes |1\rangle = |3\rangle_3$

$S_{ij}$ swaps $i, j$ bits

$C_{ij}$ controlled not

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$

---

$\mathbb{C}$ VS $(\mathbb{C}^{2^n})$

If $z = x + iy$

$\bar{z} = x - iy$

modulus / abs value

$|z| = \sqrt{z\bar{z}} = \sqrt{x^2 + y^2}$

If $\vec{u} \in \mathbb{C}^n$, $\vec{u} = (u_1, \ldots, u_n)$,

then $\overline{\vec{u}} = (\bar{u}_1, \ldots, \bar{u}_n)$

If $A$ is a matrix, define the <u>conjugate transpose</u>

$A^\dagger = (\bar{A})^T$

---

Dfn: A linear operator $L: \mathbb{C}^n \to \mathbb{C}^m$ is a fn that satisfies

- $L(\vec{u} + \vec{v}) = L(\vec{u}) + L(\vec{v})$
- $L(c\vec{u}) = cL(\vec{u})$

encoded as a $m \times n$ matrix

Dfn: $\vec{u} = (u_1, \dots, u_n)$ $\in \mathbb{C}^n$
$\vec{v} = (v_1, \dots, v_n)$

The inner product is

$$\vec{u} \cdot \vec{v} = \langle \vec{u}, \vec{v} \rangle = \langle \vec{u} | \vec{v} \rangle$$
$$= \sum_{i=1}^{n} u_i \bar{v}_i$$
$$= u_1 \bar{v}_1 + u_2 \bar{v}_2 + \dots + u_n \bar{v}_n .$$

The norm of $\vec{u}$ is

$$\| \vec{u} \| = \sqrt{\langle \vec{u} | \vec{u} \rangle}$$
$$= \sqrt{\sum_{i=1}^{n} u_i \bar{u}_i}$$
$$= \sqrt{\sum_{i=1}^{n} |u_i|^2}$$

1) Pos Def

$\langle \vec{u} | \vec{u} \rangle \geq 0$

if $\langle \vec{u} | \vec{u} \rangle = 0$, then $\vec{u} = \vec{0}$.

2) Conjugate symmetry

$\langle \vec{u} | \vec{v} \rangle = \overline{\langle \vec{v} | \vec{u} \rangle}$

3) Linear in 1st coord

$\langle \vec{u} + \vec{v} | \vec{w} \rangle = \langle \vec{u} | \vec{w} \rangle + \langle \vec{v} | \vec{w} \rangle$

$\langle c\vec{u} | \vec{w} \rangle = c \langle \vec{u} | \vec{w} \rangle$

---

Sesquilinear

$\langle \vec{u} | \vec{v} + \vec{w} \rangle = \overline{\langle \vec{v} + \vec{w} | \vec{u} \rangle} = \overline{\langle \vec{v} | \vec{u} \rangle} + \overline{\langle \vec{w} | \vec{u} \rangle} = \langle \vec{u} | \vec{v} \rangle + \langle \vec{u} | \vec{w} \rangle$

$\langle \vec{u} | c\vec{w} \rangle = \overline{\langle c\vec{w} | \vec{u} \rangle} = \bar{c} \overline{\langle \vec{w} | \vec{u} \rangle} = \bar{c} \langle \vec{u} | \vec{w} \rangle$

$\left\langle \begin{bmatrix} 2+i \\ 3-i \end{bmatrix} \middle| \begin{bmatrix} 1+i \\ 1+i \end{bmatrix} \right\rangle = \overline{(1+i)} \left\langle \begin{bmatrix} 2+i \\ 3-i \end{bmatrix} \middle| \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\rangle$

$\left\langle \begin{bmatrix} 2+i \\ 3-i \end{bmatrix} \middle| \begin{bmatrix} 1+i \\ 1+i \end{bmatrix} \right\rangle =$

$(2+i)(1-i) + (3-i)(1-i) =$

$2+i-2i+1 + 3-i-3i-1$

$= 5-5i = 5(1-i).$

Fact. If $A \in M_{m\times n}$,

$\exists!\ A^\dagger \in M_{n\times m}$ s.t

$\langle A\vec{u} \mid \vec{v}\rangle = \langle \vec{u} \mid A^\dagger \vec{v}\rangle$

for all $\vec{u} \in \mathbb{C}^n$, $\vec{v} \in \mathbb{C}^m$

call $A^\dagger$ the <u>adjoint</u> of $A$.

---

Given $\vec{u} \in \mathbb{C}^n$, define a function

$\langle \vec{u}| : \mathbb{C}^n \to \mathbb{C}$

$\vec{v} \mapsto \langle \vec{u} | \vec{v}\rangle$

vectors are kets $|\psi\rangle$ or $|\varphi\rangle$

Linear functionals $\langle\psi|$ or $|\varphi\rangle$

are bra

innerproduct $\langle \psi | \varphi\rangle$

is a bra[c]ket.

Qubits

Classical bit: $|0\rangle$ or $|1\rangle$

Quantum: can have superposition.

Dfn: a qubit is a unit vector in the span of $\{|0\rangle, |1\rangle\}$.

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$$

where $|\alpha_0|^2 + |\alpha_1|^2 = 1$.

amplitudes $\alpha_0, \alpha_1$.

Can have multi bit superpositions

$$|\psi\rangle = \sum_{0 \le x < 2^n} \alpha_x |x\rangle_n$$

where

$$\sum_{0 \le x < 2^n} |\alpha_x|^2 = 1.$$

$|5\rangle_4 = |0101\rangle$

---

2 individual qubits, can combine

$$|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$$

$$|\phi\rangle = \beta_0 |0\rangle + \beta_1 |1\rangle$$

$$|\Psi\rangle = |\psi\rangle \otimes |\phi\rangle = (\alpha_0|0\rangle + \alpha_1|1\rangle) \otimes (\beta_0|0\rangle + \beta_1|1\rangle)$$

$$= \alpha_0\beta_0|00\rangle + \alpha_0\beta_1|01\rangle + \alpha_1\beta_0|10\rangle + \alpha_1\beta_1|11\rangle \longleftrightarrow \begin{bmatrix} \alpha_0\beta_0 \\ \alpha_0\beta_1 \\ \alpha_1\beta_0 \\ \alpha_1\beta_1 \end{bmatrix}$$

not all 2-qubit states!

have $\gamma_0\gamma_3 = \gamma_1\gamma_2$ Can't always factor into single qubits.

---

If we can't factor, call state entangled.

Qubit gates

lots of reversible operations on $\mathbb{C}^2$.

But not all work.

$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$ is invertible

but doesn't send unit vectors to unit vectors.

Dfn: a square matrix is unitary if $UU^\dagger = U^\dagger U = I$.

Prop: Let $U$ be unitary. Then

1) $|\det U| = 1$

2) $\langle U\varphi | U\psi \rangle = \langle \varphi | \psi \rangle$

3) $\| U\varphi \| = \| \varphi \|$

---

Pf/ 1) $\det(U)\det(U^\dagger) = 1$

$\det(U)\overline{\det(U)} = 1$

$|\det(U)|^2 = 1$.

2) Excercise

3) $\| U\varphi \| = \sqrt{\langle U\varphi | U\varphi \rangle}$

$= \sqrt{\langle \varphi | \varphi \rangle} = \| \varphi \|$.

Any reversible classical operation gives a unitary operator.

$Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

are unitary, and thus operators on 1 qubit.

---

$\wedge, \overline{\wedge}$ not qubit operators.

a unitary matrix $\in M_{2^n}$
is an n-qubit gate.

practically speaking:
can't build arbitrary
large operators.

Want operators built
out of 1-qubit
and 2-qubit gates.

This doesn't
lose much.

# Measurement

<u>Cannot</u> detect amplitudes of a superposition.

<u>Can</u> "make a measurement".

if we measure $\alpha_0|0\rangle + \alpha_1|1\rangle$
we get either 0 w/ prob $|\alpha_0|^2$
or <u>1</u> w/ prob $|\alpha_1|^2$

Born Rule: if

$$|\psi\rangle_n = \sum_{0 \le x < 2^n} \alpha_x |x\rangle_n$$

then the prob of getting $x$ as output
is $|\alpha_x|^2$

Myth: QC powerful b/c they act on all possible inputs at once.

False.

we can operate on superposition of all possible inputs

But we can't read all outputs; they collapse when we measure.

Trick is to arrange good cancellation of 'bad' outputs