

Math 4981 Spring 2021
Cryptography HW 12 Solutions
Due Thursday, April 15

1. Solve the following knapsack problems:

- (a) $\mathbf{M} = (3, 7, 19, 43, 89, 195), S = 260$
- (b) $\mathbf{M} = (5, 11, 25, 61, 125, 261), S = 402$
- (c) $\mathbf{M} = (4, 12, 15, 36, 75, 162), S = 214$

Solution:

- (a) $260 = 195 + 43 + 19 + 3$
- (b) $402 = 261 + 125 + 11 + 5$.
- (c) $214 = 162 + 36 + 12 + 4$. Note this isn't superincreasing so the algorithm doesn't work; you have to use trial and error.

2. Explicitly use the collision algorithm, showing all steps, to solve the knapsack problem for $M = (4, 5, 11, 24, 29, 36, 39, 46)$ and $S = 88$.

3. Alice publishes the public key $\mathbf{M} = (18, 89, 90, 110, 185, 141)$

- (a) Suppose you wish to send the message $\mathbf{x} = (1, 1, 0, 1, 1, 0)$ (corresponding to 27 in binary). What ciphertext should you send?
- (b) Suppose you intercept someone else's message of $S = 430$. Express the problem of decrypting this message as a shortest-vector problem, as in section 5.2.4 of the notes.
- (c) Now decrypt the message corresponding to $S = 430$. You don't need to use lattice methods to do this.

Solution:

- (a) We have $S = 18 + 89 + 110 + 185 = 402$.
- (b) Let L be the lattice generated by

$$\begin{aligned} \mathbf{v}_1 &= (2, 0, 0, 0, 0, 0, 18) & \mathbf{v}_2 &= (0, 2, 0, 0, 0, 0, 89) \\ \mathbf{v}_3 &= (0, 0, 2, 0, 0, 0, 90) & \mathbf{v}_4 &= (0, 0, 0, 2, 0, 0, 110) \\ \mathbf{v}_5 &= (0, 0, 0, 0, 2, 0, 185) & \mathbf{v}_6 &= (0, 0, 0, 0, 0, 2, 141) \\ \mathbf{v}_7 &= (1, 1, 1, 1, 1, 1, 430). \end{aligned}$$

Then we wish to find the shortest vector in L .

- (c) Trial and error shows that $S = 89 + 90 + 110 + 141$ so the message is $\mathbf{x} = (0, 1, 1, 1, 0, 1) = 46$.

4. Alice chooses the superincreasing sequence

$$\mathbf{r} = (2, 5, 13, 28, 60, 144)$$

with the numbers $A = 53$ and $B = 300$.

- (a) What public key does Alice publish?
(b) Alice receives the ciphertext $S = 681$. What is the plaintext message?

Solution:

- (a) We have $A = 53$ so $A^{-1} \equiv 17 \pmod{300}$. Then we have

$$\mathbf{M} = (106, 265, 89, 284, 180, 132).$$

- (b) We compute $A^{-1}S = 17 \cdot 681 = 11577 \equiv 177$. Thus she sees the solution is $\mathbf{x} = (0, 1, 0, 1, 0, 1)$.

5. Suppose Alice's public key for a knapsack cryptosystem is

$$\mathbf{M} = (5186, 2779, 5955, 2307, 6599, 6771, 6296, 7306, 4115, 190).$$

Eve intercepts the encrypted message $S = 26560$. She also manages to steal from Alice the secret numbers $A = 4392$ and $B = 8387$. Use this information to find Alice's superincreasing sequence \mathbf{r} and then decrypt the message.

Solution: We have $A^{-1} \equiv 2683 \pmod{8387}$, so we have

$$\mathbf{r} = A^{-1}\mathbf{M} = (5, 14, 30, 75, 160, 351, 750, 1579, 3253, 6550).$$

To decrypt we calculate $S' = A^{-1}S = 2683 \cdot 26560 \equiv 4528 \pmod{8387}$. Then we write

$$S' = 3253 + 750 + 351 + 160 + 14.$$

Thus the message is $(0, 1, 0, 0, 1, 1, 1, 0, 1, 0) = 314$ in decimal (or 370 if you read it backwards).

6. Which of the following matrices are invertible over \mathbb{Z} ? Find inverses for the ones that are.

(a) $\begin{bmatrix} 3 & 1 \\ 2 & 2 \end{bmatrix}$

(b) $\begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}$

(c) $\begin{bmatrix} 3 & 2 & 2 \\ 2 & 1 & 2 \\ -1 & 3 & 1 \end{bmatrix}$

$$(d) \begin{bmatrix} -3 & -1 & 2 \\ 1 & -3 & -1 \\ 3 & 0 & -2 \end{bmatrix}$$

Solution:

- (a) The determinant is $6 - 2 = 4$ so the matrix is not invertible over \mathbb{Z} .
 (b) The determinant is $-3 + 4$ so the matrix is invertible over \mathbb{Z} . The inverse is

$$\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}.$$

- (c) The determinant is -9 so the matrix is not invertible over \mathbb{Z} .
 (d) The determinant is 1 so the matrix is invertible over \mathbb{Z} . The inverse is

$$\begin{bmatrix} 6 & -2 & 7 \\ -1 & 0 & -1 \\ 9 & -3 & 100 \end{bmatrix}$$

7. Let L be the lattice generated by the basis $B = \{(3, 1, -2), (1, -3, 5), (4, 2, 1)\}$. Which of the following sets of vectors are also bases for L ? For each one that is, find the change of basis matrix and write the new basis in terms of the basis B .

- (a) $B_1 = \{(5, 13, -13), (0, -4, 2), (-7, -13, 18)\}$
 (b) $B_2 = \{(4, -2, 3), (6, 6, -6), (-2, -4, 7)\}$.

Solution:

- (a) We have $\mathbf{w}_1 = -3\mathbf{v}_2 + 2\mathbf{v}_3$, $\mathbf{w}_2 = \mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_3$, and $\mathbf{w}_3 = -2\mathbf{v}_1 + 3\mathbf{v}_2 - \mathbf{v}_3$. Thus we have the matrix

$$U = \begin{bmatrix} 0 & -3 & 2 \\ 1 & 1 & -1 \\ -2 & 3 & -1 \end{bmatrix}$$

We have $\det U = 1$ so this is a basis.

- (b) We have

$$\begin{aligned} \mathbf{w}_1 &= \mathbf{v}_1 + \mathbf{v}_2 \\ \mathbf{w}_2 &= \mathbf{v}_1 - \mathbf{v}_2 + \mathbf{v}_3 \\ \mathbf{w}_3 &= -\mathbf{v}_1 + \mathbf{v}_2 \end{aligned}$$

Then we have

$$U = \begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

and $\det U = -2$ so this isn't a basis.

8. Let $L_1 \subset \mathbb{R}^2$ be generated by $B_1 = \{(1, 3), (-1, 2)\}$ and let $L_2 \subset \mathbb{R}^2$ be generated by $B_2 = \{(2, 4), (3, -1)\}$. Sketch the fundamental domains $\mathcal{F}(B_1)$ and $\mathcal{F}(B_2)$. What are the areas of these domains?

9. A lattice L has dimension $n = 251$ and determinant $\det(L) \approx 2^{2251.58}$. How long do you expect the shortest vector to be?

Solution: By the Gaussian heuristic, we estimate the shortest length to probably be

$$\sigma(L) = \sqrt{\frac{251}{2\pi e}} (2^{2251.58})^{1/251} \approx 1923.$$