Math 1231 Practice Midterm Solutions

Instructor: Jay Daigle

1. These are the instructions you will see on the real test, next week. I include them here so you know what to expect.
2. You will have 75 minutes for this test.
3. You are not allowed to consult books or notes during the test, but you may use a one-page, one-sided, handwritten cheat sheet you have made for yourself ahead of time. You may *not* use a calculator.

Name:

Recitation Section:

Problem 1 (M1)**.** Compute the following limits if they exist. Show enough work to justify your computation, or your claim that the limit does not exist.

(a)

$$\lim_{x\to 9} \frac{3-\sqrt{x}}{9-x}$$

Solution:

$$\lim_{x\to 9} \frac{3-\sqrt{x}}{9-x} = \lim_{x\to 9} \frac{(3-\sqrt{x})(3+\sqrt{x})}{(9-x)(3+\sqrt{x})} = \lim_{x\to 9} \frac{9-x}{(9-x)(3+\sqrt{x})} = \lim_{x\to 9} \frac{1}{3+\sqrt{x}} = 1/6.$$

(b)

$$\lim_{x\to -\infty} \frac{3x^3+\sqrt[3]{x}}{\sqrt{9x^6+2x^2+1}+x}$$

Solution:

$$\begin{aligned}\lim_{x\to -\infty} \frac{3x^3+\sqrt[3]{x}}{\sqrt{9x^6+2x^2+1}+x} &= \lim_{x\to -\infty} \frac{3x^3/x^3+\sqrt[3]{x}/x^3}{\sqrt{9x^6+2x^2+1}/(-\sqrt{x^6})+x/x^3} \\ &= \lim_{x\to -\infty} \frac{3+x^{-8/3}}{-\sqrt{9+2x^{-4}+x^{-6}}+x^{-2}} \\ &= \lim_{x\to -\infty} \frac{3}{-\sqrt{9}} = -1.\end{aligned}$$

(c)

$$\lim_{x\to 1} \frac{\sin^2(x-1)}{(x-1)^2} =$$

Solution:

$$\lim_{x\to 1} \frac{\sin^2(x-1)}{(x-1)^2} = \lim_{x\to 1} \left(\frac{\sin(x-1)}{x-1}\right)^2 = \left(\lim_{x\to 1} \frac{\sin(x-1)}{x-1}\right)^2 = 1^2 = 1$$

by the small angle approximation.

(d)

$$\lim_{x\to 3} \frac{x-5}{(x-3)^2} =$$

Solution:

$$\lim_{x\to 3} \frac{x-5}{(x-3)^2} = -\infty$$

since the top approaches -2 and the bottom approaches zero and is always positive.

Problem 2 (M2)**.** Compute the derivatives of the following functions using methods we have learned in class. Show enough work to justify your answers.

(a) $f(x) = \sec\left(\frac{\sqrt{x^2+1}}{x+2}\right)$

Solution:

$$f'(x) = \sec\left(\frac{\sqrt{x^2+1}}{x+2}\right) \cdot \tan\left(\frac{\sqrt{x^2+1}}{x+2}\right) \cdot \frac{\frac{1}{2}(x^2+1)^{-1/2}2x(x+2) - \sqrt{x^2+1}}{(x+2)^2}$$

(b) $g(x) = \sqrt[4]{\frac{x^3+\cos(x^2)}{\sin(x^3)+1}}$

Solution:

$$g'(x) = \frac{1}{4}\left(\frac{x^3+\cos(x^2)}{\sin(x^3)+1}\right)^{-3/4} \cdot \frac{(3x^2 - \sin(x^2)2x)(\sin(x^3)+1) - \cos(x^3)3x^2(x^3+\cos(x^2))}{(\sin(x^3)+1)^2}$$

Problem 3 (S1)**.**
Suppose $f(x) = x^2 - 6x$, and we want an output of approximately -9. What input a should we aim for? Find a δ so that if our input is $a \pm \delta$ then our output will be -9 ± 2. Justify your answer.

Solution: We want an input of about $a = 3$. Our output error will be $|x^2 - 6x + 9| = |x-3|^2$. We want this to be less than 2, so we need

$$\begin{aligned} |x-3|^2 &< 2 \\ |x-3| &< \sqrt{2}, \end{aligned}$$

so we can take $\delta = \sqrt{2}$.

Problem 4 (S2)**.** Show that $\lim_{x\to 0} x \sin\left(\frac{3}{x}\right) = 0$.

Solution: We know that

$$\begin{aligned} -1 \le \sin\left(\frac{3}{x}\right) &\le 1 \\ -|x| \le x\sin\left(\frac{3}{x}\right) &\le |x| \end{aligned}$$

Since $\lim_{x\to 0} -|x| = 0$ and $\lim_{x\to 0} |x| = 0$, by the Squeeze Theorem, we know that $\lim_{x\to 0} x\sin\left(\frac{3}{x}\right) = 0$.

Problem 5 (S3)**. Directly from the definition of derivative,** compute the derivative of $f(x) = x^2 + \sqrt{x}$ at $a = 2$.

Solution:

$$\begin{aligned} f'(2) &= \lim_{h\to 0} \frac{f(2+h) - f(2)}{h} \\ &= \lim_{h\to 0} \frac{(2+h)^2 + \sqrt{2+h} - 2^2 - \sqrt{2}}{h} \\ &= \left(\lim_{h\to 0} \frac{4h+h^2}{h}\right) + \left(\lim_{h\to 0} \frac{(\sqrt{2+h}-\sqrt{2})(\sqrt{2+h}+\sqrt{2})}{h(\sqrt{2+h}+\sqrt{2})}\right) \\ &= \left(\lim_{h\to 0} 4+h\right) + \left(\lim_{h\to 0} \frac{1}{\sqrt{2+h}+\sqrt{2}}\right) \\ &= 4 + \frac{1}{2\sqrt{2}}. \end{aligned}$$