Math 1231 Practice Midterm 2 Solutions

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(a) These are the instructions you will see on the real test, next week. I include them here so you know what to expect.

(b) You will have 75 minutes for this test.

(c) You are not allowed to consult books or notes during the test, but you may use a one-page, one-sided, handwritten cheat sheet you have made for yourself ahead of time.

(d) You may not use a calculator. You may leave answers unsimplified, except you should compute trigonometric functions as far as possible.

(e) The exam has 5 problems, one on each mastery topic since the first midterm. The exam has 4 pages total.

(f) Read the questions carefully and make sure to answer the actual question asked. Make sure to justify your answers—math is largely about clear communication and argument, so an unjustified answer is much like no answer at all.

When in doubt, show more work and write complete sentences.

(g) If you need more paper to show work, I have extra at the front of the room.

(h) Good luck!

Name:

Recitation Section:

Problem 1 (M3)**.**

(a) Find a tangent line to the curve given by $x^4 - 2x^2y^2 + y^4 = 16$ at the point $(\sqrt{5}, 1)$.

Solution: We use implicit differentiation, and find that

$$4x^3 - 2\left((2xy^2 + x^2 2y\frac{dy}{dx}\right) + 4y^3\frac{dy}{dx} = 0$$

$$4x^3 - 4xy^2 = 4x^2y\frac{dy}{dx} - 4y^3\frac{dy}{dx}$$

$$\frac{4x^3 - 4xy^2}{4x^2y - 4y^3} = \frac{dy}{dx}$$

Thus at the point $(\sqrt{5}, 1)$ we have

$$\frac{dy}{dx} = \frac{4\sqrt{5}^3 - 4\sqrt{5}\cdot 1^2}{4\sqrt{5}^2\cdot 1 - 4\cdot 1^3} = \sqrt{5}\left(\frac{20-4}{20-4}\right) = \sqrt{5}.$$

Thus the equation of our tangent line is

$$y - y_0 = m(x - x_0)$$
$$y - 1 = \sqrt{5}(x - \sqrt{5}).$$

(b) Give equation for the linear approximation of the function $f(x) = x\sin(x)$ near the point $a = \pi/2$. Use it to estimate $f(1.5)$.

Solution: We calculate that $f(\pi/2) = \pi/2\sin(\pi/2) = \pi/2$, and $f'(x) = \sin(x) + x\cos(x)$, so $f'(\pi/2) = \sin(\pi/2) + \pi/2\cos(\pi/2) = 1$. So

$$f(x) \approx \pi/2 + 1(x - \pi/2) = x.$$

Thus we have

$$f(1.5) \approx 1.5.$$

(The true answer is $1.49624\ldots$.)

Problem 2 (M4)**.**

(a) Find and classify all the critical points of $f(x) = (x-5)\sqrt[3]{x^2}$.

Solution: We compute

$$f'(x) = \sqrt[3]{x^2} + (x-5)\frac{2}{3}x^{-1/3} = x^{2/3} + \frac{2x-10}{3\sqrt[3]{x}}$$
$$= \frac{3x + 2x - 10}{3\sqrt[3]{x}} = 5\frac{x-2}{3\sqrt[3]{x}}$$

This is equal to zero when $x = 2$ and is undefined when $x = 0$, so the two critical points are $x = 0$ and $x = 2$.

We could try to use the second derivative test here, but it won't really work. We get

$$f''(x) = \frac{10x + 10}{9x^{4/3}}$$
$$f''(2) = \frac{5}{3\sqrt[3]{2}} > 0$$

so we see that f has a local minimum at $x = 2$, but $f''(0)$ is undefined so it doesn't help us classify $x = 0$.

Instead we compute a chart

	$5(x-2)$	$\frac{1}{3\sqrt[3]{x}}$	$f'(x)$
$x < 0$	$-$	$-$	$+$
$0 < x < 2$	$-$	$+$	$-$
$2 < x$	$+$	$+$	$+$

Thus we conclude that f has a local maximum at $x = 0$ and a local minimum at $x = 2$.

(b) Find the absolute extrema of $g(x) = (x^2 - 3x)\sqrt[3]{x-3}$ on $[1, 4]$, and justify your claim that these are the absolute extrema.

Solution: We compute

$$g'(x) = (2x-3)\sqrt[3]{x-3} + (x^2 - 3x)\frac{1}{3}(x-3)^{-2/3}.$$

This is undefined when $x = 3$, and to find the zeroes we compute

$$\begin{aligned}
0 &= (2x-3)\sqrt[3]{x-3} + (x^2 - 3x)\frac{1}{3}(x-3)^{-2/3} \\
&= (2x-3)\sqrt[3]{x-3} + \frac{x^2 - 3x}{3(x-3)^{2/3}} \\
0 &= (2x-3)\sqrt[3]{x-3} \cdot 3(x-3)^{2/3} + (x^2 - 3x) \\
&= (2x-3)3(x-3) + (x^2 - 3x) = 3(2x^2 - 3x - 6x + 9) + x^2 - 3x \\
&= 7x^2 - 30x + 27 = (7x - 9)(x - 3)
\end{aligned}$$

which is zero when $x = 3$ or $x = 9/7$. So then we have

$$\begin{aligned}
g(1) &= -2\sqrt[3]{-2} = 2\sqrt[3]{2} \\
g(9/7) &= \left(\frac{91}{49} - \frac{27}{7}\right)\sqrt[3]{-12/7} = \frac{108}{49}\sqrt[3]{12/7} \\
g(3) &= 0 \\
g(4) &= 4 \cdot \sqrt[3]{1} = 4.
\end{aligned}$$

All of these numbers are non-negative, so the minimum value is 0, at $x = 3$. We can see that $2\sqrt[3]{2} < 2 \cdot 2 = 4$, and maybe convince ourselves that

$$\frac{108}{49}\sqrt[3]{12/7} \approx 2 \cdot 1 < 4.$$

Thus the maximum value is 4, which occurs at 4.

Problem 3 (S4)**.** Suppose that $Q(p) = 3p^2 + 10p - 100$ is the number of widgets you can buy at a price of p dollars.

(i) What are the units of $Q'(p)$? What does it represent physically? What does it mean if $Q'(p)$ is big?

Solution: $Q'(p)$ has units of widgets per dollar. The derivative is the rate at which increasing the price increases the number of widgets you can buy (called the marginal elasticity of demand, though you don't need to know that on the test). If $Q'(p)$ is big, that means that raising your price by 1 dollar gets you a lot more widgets available to buy.

(ii) Calculate $Q'(10)$. What does this tell you physically? What physical observation could you make to check your calculation?

Solution: $Q'(p) = 6p + 10$ so $Q'(10) = 70$. This means that if you are buying widgets for \$10, you can get approximately seventy more widgets if you raise your price to \$11.

Problem 4 (S5)**.** The surface area of a cube is given by the formula $A = 6s^2$ where s is the length of a side. If the side lengths are increasing by 2 inches per second, how fast is the surface area increasing when the area is 54 square inches?

Solution: We have the data $A = 6s^2, A = 54, s' = 2$. We take a derivative and see that $A' = 12ss'$, so we need to find s. But when $A = 54$ we have

$$\begin{aligned} 54 &= 6s^2 \\ 9 &= s^2 \\ 3 &= s \end{aligned}$$

and thus

$$A' = 12ss' = 12 \cdot 3 \cdot 2 = 72$$

so the area is increasing at 72 square inches per second.

Problem 5 (S6)**.** Let $f(x) = \dfrac{x^3 - 2}{x^4}$. We compute that $f'(x) = \dfrac{8 - x^3}{x^5}$ and $f''(x) = \dfrac{2x^3 - 40}{x^6}$. Sketch a graph of f.

Your answer should discuss the domain, asymptotes, roots, limits at infinity, critical points and values, intervals of increase and decrease, and concavity.

Solution: The function is defined everywhere except at 0. Near zero, we can see the top is always negative and the bottom is always positive, so $\lim_{x\to 0} f(x) = -\infty$ and we should have a downwards asymptote on either side.

We see there is a root at $x = \sqrt[3]{2}$, and $\lim_{x\to\pm\infty} f(x) = 0$.

We see that $f'(x)$ is undefined at $x = 0$, and is zero when $x^3 = 8$ and thus when $x = 2$. So our critical points occur at 0 and 2. We calculate $f(2) = \frac{6}{16}$, and f isn't defined at 0. By making a chart, we get

	$8 - x^3$	x^5	$f'(x)$
$x < 0$	$+$	$-$	$-$
$0 < x < 2$	$+$	$+$	$+$
$2 < x$	$-$	$+$	$-$

so f is decreasing for values less than zero or greater than 2, and increasing for values between 0 and 2.

The second derivative is undefined at 0, and is zero when $2x^3 - 40$ and so when $x = \sqrt[3]{20}$, so our potential points of inflection are $0, \sqrt[3]{20}$. We compute $f(\sqrt[3]{20}) = \frac{18}{20\sqrt[3]{20}}$. We can make a chart again, but we see that the denominator of $f''(x) \geq 0$, so $f''(x) > 0$ if $x > \sqrt[3]{20}$ and $f''(x) < 0$ if $x < \sqrt[3]{20}$.